BOOK IV

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CHAPTER III[1]

NUMERICAL VALUES OF THE PERTURBATIONS

885. IT is known by observation that the sidereal revolutions of the satellites are accomplished in the following periods:–

	Days
1st Satellite in	1.769137787
2nd Satellite in	3.551181017
3rd Satellite in	7.154552808
4th Satellite in	16.689019396

The values of n, n_1, n_2, n_3, being reciprocally as these periods,

$$n = n_3 . 9.433419$$
$$n_1 = n_3 . 4.699569$$
$$n_2 = n_3 . 2.332643 .$$

And as the sidereal revolution of Jupiter is 4332.602208 days,

$$M = n_3 . 00385196.$$

886. The mean distances of the satellites from Jupiter are known from observation; with them, by a method to be shown afterwards, the equations (271) and (290) give the following approximate values of the masses of the satellites, and of the compression of Jupiter

$$m = 0.0000184113$$
$$m_1 = 0.0000258325$$
$$m_2 = 0.0000865185$$
$$m_3 = 0.00005590808$$
$$r - \tfrac{1}{2} f = 0.0217794,$$

the mass of Jupiter being the unit.

887. The mean distances of the three first satellites cannot be measured with sufficient accuracy for computing the inequalities; it is therefore necessary to determine them from the value of a_3 by Kepler's law.

At the mean distance of Jupiter from the sun, his equatorial diameter is seen under an angle of 38″.99; taking this diameter as the unit, the mean distance of the fourth satellite in functions of the diameter is

$$a_3 = 25.43590.$$

By article 818 the mean distance of a satellite is $a + da$, in consequence of the action of the disturbing forces; but as the variation da is principally owing to the compression of Jupiter, the only part of da in article 821 that has a sensible effect on the mean distances is $a\frac{\left(r - \frac{1}{2}f\right)}{3a^2}$, hence $a = n^{-\frac{2}{3}}$ becomes

$$a = n^{-\frac{2}{3}}\left(1 + \tfrac{1}{3}\left(\frac{r - \frac{1}{2}f}{a^2}\right)\right),$$

also

$$a_3 = n_3^{-\frac{2}{3}}\left(1 + \tfrac{1}{3}\left(\frac{r - \frac{1}{2}f}{a_3^2}\right)\right);$$

and thus, by Kepler's law,

$$a = \left\{1 + \tfrac{1}{3}\left(r - \tfrac{1}{2}f\right)\left(\frac{1}{a^2} - \frac{1}{a_3^2}\right)\right\} a_3 \sqrt[3]{\frac{n^2}{n_3^2}}$$

in which

$$\frac{1}{a^2} = \frac{1}{\left(a_3 \sqrt[3]{\frac{n^2}{n_3^2}}\right)^2};$$

whence, with the preceding values, it is easy to find that

$$\begin{aligned} a &= 5.698491 \\ a_1 &= 9.066548 \\ a_2 &= 14.461893 \\ a_3 &= 25.43590\,; \end{aligned}$$

with these the series S and S' in article 453 may be computed, and from them all the coefficients A_0, A_1, &c.; B_0, B_1, &c.; and their differences may be found by the same method of computation, and from the same formulae, as for the planets; and thence

$$N = n_3 . 9.4269167$$

$$N_1 = n_3 \,.\, 4.6979499$$
$$N_2 = n_3 \,.\, 2.332309$$
$$N_3 = n_3 \,.\, 0.9999070.$$

888. With these quantities, the perturbations in longitude and distance computed from the expressions in articles 820 and 821 are

(296)

$$d\,v = +m_1 \left\{ \begin{array}{l} + \quad 60''.7333 \sin\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - 7042''.63 \sin 2\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 22''.949 \sin 3\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 5''.2464 \sin 4\{n_1 t + nt + \epsilon_1 - \epsilon\} \\ - \quad 1''.7518 \sin 5\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0''.69443 \sin 6\{n_1 t - nt + \epsilon_1 - \epsilon\} \end{array} \right\}$$
$$+m_2 \left\{ \begin{array}{l} + \quad 7''.1065 \sin\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 6''.0005 \sin 2\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 0''.6162 \sin 3\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 0''.1156 \sin 4\{n_2 t - nt + \epsilon_2 - \epsilon\} \end{array} \right\}$$
$$+ \quad 0''.04731 \sin\{2nt - Mt + 2\epsilon - \epsilon E\}$$

The inequalities depending on m_3 are insensible.

$$d\,r = +m_1 \left\{ \begin{array}{l} + \quad 0.000084865 \\ + \quad 0.00046652 \cos\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0.09764199 \cos 2\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0.00040917 \cos 3\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0.00010761 \cos 4\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0.00003824 \cos 5\{n_1 t - nt + \epsilon_1 - \epsilon\} \\ - \quad 0.00001642 \cos 6\{n_1 t - nt + \epsilon_1 - \epsilon\} \end{array} \right\}$$
$$+m_2 \left\{ \begin{array}{l} + \quad 0.00000703 \\ + \quad 0.00007780 \cos\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 0.00010631 \cos 2\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 0.00001310 \cos 3\{n_2 t - nt + \epsilon_2 - \epsilon\} \\ - \quad 0.00000269 \cos 4\{n_2 t - nt + \epsilon_2 - \epsilon\} \end{array} \right\}$$

$$+m_3\left\{\begin{array}{ll} + & 0.00000113 \\ + & 0.00001478\cos\{n_3t-nt+\in_3-\in\} \\ - & 0.00000968\cos 2\{n_3t-nt+\in_3-\in\} \\ - & 0.00000078\cos 3\{n_3t-nt+\in_3-\in\} \end{array}\right\}$$

$$\begin{array}{ll} + & 0.00000095 \\ - & 0.00000095\cos\{2Mt-nt+2E-2\in\} \end{array}$$

(297)

$$d\,v_{/} = +m\left\{\begin{array}{ll} - & 2252''.28\sin\{nt-n_1t+\in-\in_1\} \\ - & 17''.053\sin 2\{nt-n_1t+\in-\in_1\} \\ - & 3''.4102\sin 3\{nt-n_1t+\in-\in_1\} \\ - & 1''.0837\sin 4\{n_1t-nt+\in-\in_1\} \\ - & 0''.4202\sin 5\{nt-n_1t+\in-\in_1\} \end{array}\right\}$$

$$+m_2\left\{\begin{array}{ll} + & 59''.784\sin\{n_2t-n_1t+\in_2-\in_1\} \\ - & 3923''.3\sin 2\{n_2t-n_1t+\in_2-\in_1\} \\ - & 22''.318\sin 3\{n_2t-n_1t+\in_2-\in_1\} \\ - & 5''.1076\sin 4\{n_2t-n_1t+\in_2-\in_1\} \\ - & 1''.7041\sin 5\{n_2t-n_1t+\in_2-\in_1\} \\ - & 0''.6744\sin 6\{n_2t-n_1t+\in_2-\in_1\} \end{array}\right\}$$

$$+m_3\left\{\begin{array}{ll} + & 4''.0098\sin\{n_3t-n_1t+\in_3-\in_1\} \\ - & 3''.5108\sin 2\{n_3t-n_1t+\in_3-\in_1\} \\ - & 0''.3449\sin 3\{n_3t-n_1t+\in_3-\in_1\} \end{array}\right\}$$

$$+ \quad 0''.1906\sin\{2n_1t-2Mt+2\in_1-2E\}$$

$$d\,r_1 = +m\left\{\begin{array}{ll} - & 0.00044608 \\ + & 0.05069318\cos\{nt-n_1t+\in-\in_1\} \\ + & 0.00059197\cos 2\{nt-n_1t+\in-\in_1\} \\ + & 0.00014002\cos 3\{nt-n_1t+\in-\in_1\} \\ + & 0.00004784\cos 4\{nt-n_1t+\in-\in_1\} \\ + & 0.00001928\cos 5\{nt-n_1t+\in-\in_1\} \end{array}\right\}$$

$$
+m_2\left\{\begin{array}{ll}
+ & 0.00006497 \\
+ & 0.00073255\cos\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\
- & 0.08670960\cos2\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\
- & 0.00063398\cos3\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\
- & 0.00016685\cos4\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\
- & 0.00006067\cos5\{n_2t-n_1t+\epsilon_2-\epsilon_1\}
\end{array}\right\}
$$

$$
+m_3\left\{\begin{array}{ll}
+ & 0.00000798 \\
+ & 0.00007146\cos\{n_3t-n_1t+\epsilon_3-\epsilon_1\} \\
- & 0.00010133\cos2\{n_3t-n_1t+\epsilon_3-\epsilon_1\} \\
- & 0.00001189\cos3\{n_3t-n_1t+\epsilon_3-\epsilon_1\}
\end{array}\right\}
$$

$$
\begin{array}{ll}
+ & 0.00000609 \\
- & 0.00000609\cos\{2Mt-2n_1t+2E-2\epsilon_1\}
\end{array}
$$

(298)

$$
dv_2=+m\left\{\begin{array}{ll}
+ & 7''.862\sin\{nt-n_2t+\epsilon-\epsilon_2\} \\
- & 0''.228\sin2\{nt-n_2t+\epsilon-\epsilon_2\} \\
- & 0''.0414\sin3\{nt-n_2t+\epsilon-\epsilon_2\}
\end{array}\right\}
$$

$$
+m_1\left\{\begin{array}{ll}
- & 1126''.96\sin\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\
- & 16''.504\sin2\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\
- & 3''.2995\sin3\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\
- & 1''.0467\sin4\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\
- & 0''.4067\sin5\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\
- & 0''.1767\sin6\{n_1t-n_2t+\epsilon_1-\epsilon_2\}
\end{array}\right\}
$$

$$
+m_3\left\{\begin{array}{ll}
+ & 34''.396\sin\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\
- & 117''.32\sin2\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\
- & 8''.251\sin3\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\
- & 1''.919\sin4\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\
- & 0''.609\sin5\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\
- & 0''.227\sin6\{n_3t-n_2t+\epsilon_3-\epsilon_2\}
\end{array}\right\}
$$

$$
+\quad 0''.7734\sin\{2n_2t-2Mt+2\epsilon_2-2E\}
$$

$$d\,r_2 = +m\begin{cases} - & 0.00054798 \\ + & 0.00059147\cos\{nt - n_2t + \in - \in_2\} \\ + & 0.00001906\cos 2\{nt - n_2t + \in - \in_2\} \\ + & 0.00000348\cos 3\{nt - n_2t + \in - \in_2\} \end{cases}$$

$$+m_1\begin{cases} - & 0.00070942 \\ + & 0.04137743\cos\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00091726\cos 2\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00021712\cos 3\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00007409\cos 4\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00002980\cos 5\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00001318\cos 6\{n_1t - n_2t + \in_1 - \in_2\} \end{cases}$$

$$+m_3\begin{cases} + & 0.00006850 \\ + & 0.00075191\cos\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.0044961\cos 2\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00039801\cos 3\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00010474\cos 4\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00003569\cos 5\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00001379\cos 6\{n_3t - n_2t + \in_3 - \in_2\} \end{cases}$$

$$\begin{matrix} + & 0.00003944 \\ - & 0.00003944\cos\{2Mt - 2n_3t + 2E - 2\in_3\} \end{matrix}$$

$$d\,v_3 = +m\begin{cases} + & 4''.6156\sin\{nt - n_3t + \in - \in_3\} \\ - & 0''.0067\sin 2\{nt - n_3t + \in - \in_3\} \end{cases} \qquad \textbf{(299)}$$

$$+m_1\begin{cases} - & 7''.2745\sin\{n_1t - n_3t + \in_1 - \in_3\} \\ - & 0''.09995\sin 2\{n_1t - n_3t + \in_1 - \in_3\} \\ - & 0''.0175\sin 3\{n_1t - n_3t + \in_1 - \in_3\} \end{cases}$$

$$+m_2\begin{cases} - & 11''.482\sin\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 5''.1701\sin 2\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 1''.0787\sin 3\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 0''.3304\sin 4\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 0''.1210\sin 5\{n_2t - n_3t + \in_2 - \in_3\} \end{cases}$$

$$+ \quad 4''.2082\sin 2\{n_3t - Mt + \in_3 - E\}$$

$$d\,r_3 = +m\begin{Bmatrix} - & 0.00088152 \\ + & 0.00057018\cos\{nt - n_3t + \in - \in_3\} \\ + & 0.00000113\cos2\{nt - n_3t + \in - \in_3\} \end{Bmatrix}$$
$$+m_1\begin{Bmatrix} - & 0.00093981 \\ + & 0.00091758\cos\{n_1t - n_3t + \in_1 - \in_3\} \\ + & 0.00001095\cos2\{n_1t - n_3t + \in_1 - \in_3\} \\ + & 0.00000166\cos3\{n_1t - n_3t + \in_1 - \in_3\} \end{Bmatrix}$$
$$+m_2\begin{Bmatrix} - & 0.00114443 \\ + & 0.00326071\cos\{n_2t - n_3t + \in_2 - \in_3\} \\ + & 0.00057836\cos2\{n_2t - n_3t + \in_2 - \in_3\} \\ + & 0.00013614\cos3\{n_2t - n_3t + \in_2 - \in_3\} \end{Bmatrix}$$
$$+ \quad 0.00037741$$
$$- \quad 0.00037741\cos\{2Mt - 2n_3t + 2E - 2\in_3\}.$$

These inequalities in the circular orbits are independent of their positions.

Determination of the Masses of the Satellites and the Compression of Jupiter

889. Approximate values of the masses of the satellites, and of the compression of Jupiter, are sufficiently accurate for calculating the periodic inequalities in the circular orbit; but it is necessary to have more correct values of these quantities for computing the secular variations. The periodic and secular inequalities determined by theory, when compared with their observed values, furnish the means of finding the true values of these very minute quantities. The principle periodic inequality in the longitude of the first satellite is, by observation, $1636''.4$ at its maximum; but by article 888 this inequality is, by theory, $7042''.6m_1$, whence

$$m_1 = 0.232355.$$

The greatest periodic inequality in the longitude of the second satellite is, by observation, $3862''.3$ at its maximum; the same, by (298), is

$$m\,.\,2252''.28 + m_2\,.\,3923''.3,$$

which arises from the combined action of the first and third satellites, hence

$$m = 1.714843 - m_2\,.\,1.741934. \qquad \textbf{(300)}$$

The other unknown quantities must be computed from equations (271) and (290). For that purpose let

$$r - \tfrac{1}{2} f = m \cdot 0.0217794,$$

m being an indeterminate quantity depending on the compression of Jupiter's spheroid. Then from the expressions

$$\frac{\left(r - \tfrac{1}{2} f\right)}{a^2} n = (0) \qquad \tfrac{3}{4} \cdot \frac{\mathrm{M}^2}{n} = \boxed{0}$$

and the formulae in article 474, it will be found that

$$\begin{array}{llll}
(0) = 179{,}459''m & \boxed{0} = & 33''.47 & \textbf{(301)} \\
(1) = 35{,}317''m & \boxed{1} = & 67''.16 & \\
(2) = 6{,}889''.6m & \boxed{2} = & 135''.31 & \\
(3) = 954''.82m & \boxed{3} = & 315''.64 & \\
(0.1) = m_1 \; 12{,}903''.6 & \boxed{0.1} = m_1 & 9{,}563''.2 & \\
(0.2) = m_2 \; 1{,}686''.44 & \boxed{0.2} = m_2 & 813''.69 & \\
(0.3) = m_2 \; 248''.57 & \boxed{0.3} = m_3 & 69''.16 & \\
(1.0) = m \; 10{,}229''.9 & \boxed{1.0} = m & 7{,}581''.6 & \\
(1.2) = m_2 \; 6{,}339''.61 & \boxed{1.2} = m_2 & 4{,}688''.2 & \\
(1.3) = m_3 \; 584''.554 & \boxed{1.3} = m_3 & 256''.12 & \\
(2.0) = m \; 1{,}058''.61 & \boxed{2.0} = m & 510''.77 & \\
(2.1) = m_1 \; 5{,}019''.6 & \boxed{2.1} = m_1 & 3{,}712''.1 & \\
(2.3) = m_3 \; 1{,}907''.34 & \boxed{2.3} = m_2 & 1{,}294''.4 & \\
(3.0) = m \; 117''.64 & \boxed{3.0} = m & 32''.74 & \\
(3.1) = m_1 \; 348''.99 & \boxed{3.1} = m_1 & 152''.93 & \\
(3.2) = m_2 \; 1{,}438''.2 & \boxed{3.2} = m_2 & 976''.01 &
\end{array}$$

The numerical values of $F, G;\ F',\ G',$ are determined from articles 825, 826, and 927, to be

$$\begin{array}{ll}
F\ = 1.483732 & G\ = -0.857159 \\
F' = 1.466380 & G' = -0.855370
\end{array}$$

and with the same quantities the coefficients $Q,\ Q_1,\ Q_2,$ of the equations in article are found to be[2]

$$Q = -m_1 \left\{ \frac{16.850204h - 6.118274h_1}{\left(1 + \frac{g}{972421''}\right)^2} \right\}$$

$$Q_1 = +m_2 \left\{ \frac{4.133080h_1 - 1.511476h_2}{\left(1 + \frac{g}{972421''}\right)^2} \right\}$$

$$+m \left\{ \frac{13.307450h - 4.831907h_1}{\left(1 + \frac{g}{972421''}\right)^2} \right\} \qquad \textbf{(302)}$$

$$Q_2 = -m_1 \left\{ \frac{3.248934h_1 - 1.188133h_2}{\left(1 + \frac{g}{972421''}\right)^2} \right\}$$

Not only these quantities, but several data from observation are requisite for the determination of the unknown quantities from equations (271) and (290).

890. The eclipses of the third satellite show it to have two distinct equations of the centre; the one depending on the apsides of the fourth satellite is $2h_2 = 245''.14$. The other datum is the equation of the centre of the fourth satellite, which is, by observation, equal to $3002''.04 = 2h_3$. Again, observation gives the annual and sidereal motion of the apsides of the fourth satellite equal to $2578''.75$, which, by article 831, is one of the four roots of g in equation (271), so that $g_3 = 2578''.75$. And, lastly, observation gives $43{,}374''$ for the annual and sidereal motion of the nodes of the orbit of the second satellite on the fixed plane, which is one of the roots of p in equation (290), so that

$$p_{/} = 43{,}374''.$$

891. If the values of $m_{/}$ and m, as well as all the quantities that precede, be substituted in equations (271) and (290), they become, when the first are divided by h_3, and the last by l_1,[3]

$$0 = 2182'' - 954''.81m - 117''.64m + 32''.73m\frac{h}{h_3} - 1358''.5m_2 + 35''.533\frac{h_1}{h_2}. \qquad \textbf{(303)}$$

$$0 = -\frac{h}{h_2}\left\{8040''.9 + 179457''m + 51581''.5m + 1686''.44m_2 + 248''.55m_3\right\}$$
$$+\left\{4977''.22 + 18729''m - 16020''m_2\right\}\frac{h_1}{h_3} + 544''.86m_2 + 69''.16m_3. \tag{304}$$

$$0 = +\left\{18305''.3m + 72999''.2m^2 - 63180''.4mm_2\right\}\frac{h_1}{h_3}$$
$$+\left\{\begin{array}{l} +2511''.6 - 35317''m - 14128m - 13455''m_2 - 584''.554m_3 \\ -26505''.7m^2 + 45344''.8mm_2 - 19393''.4m_2^2 \end{array}\right\}\frac{h_1}{h_3} \tag{305}$$
$$+594''.41m + 256''.12\,m_3 - 677''.04mm_2 + 592''.6m_2^2.$$

$$0 = +4831''.9m\frac{h}{h_3} + \left\{1352''.8 - 1569''m + 1342''m_2\right\}\frac{h_1}{h_3}$$
$$+89''.7 - 562''.6m - 86''.44m - 40''m_2 + 1138''.7m_2. \tag{306}$$

$$0 = +43306''.9 - 35317''m - 10229''.9m\left(1 - \frac{l}{l_/}\right)$$
$$-6339''.6m_2\left(1 - \frac{l_2}{l_1}\right) - 584''.554m_3\left(1 - \frac{l_3}{l_1}\right). \tag{307}$$

$$0 = +2998''.23 + \left(40342''.3 - 179457''m - 1686''.44m_2 - 248''.57m_3\right)\frac{l}{l_1}$$
$$+1686''.44m_2\frac{l_2}{l_1} + 248''.57m_3\frac{l_3}{l_1}. \tag{308}$$

$$0 = +1166''.5 + 1058''.6m\frac{l}{l_1} + 1907''.34m_3\frac{l_3}{l_1}$$
$$+\left\{42072''.4 - 6889''.6m - 1058''.6m - 1907''.35m_3\right\}\frac{l_2}{l_1}. \tag{309}$$

$$0 = +81''.09 + 117''.64m\frac{l}{l_1} + 1438''.2m_2\frac{l_2}{l_1}$$
$$+\left\{42976''.3 - 954''.82m - 117''.64m - 1438''.2m_2\right\}\frac{l_3}{l_1}. \tag{310}$$

892. These are the particular values of equations (271) and (290) corresponding to the roots[4] $g_3 = 2578''.82$ and $p_1 = 43374''$ alone. By the following method of approximation, nine of the unknown quantities are obtained from these eight equations, together with equation (300).

The inclinations of the satellites are very small, and the two first move nearly in circular orbits, therefore the quantities

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

are so minute, that they may be made zero in the equations (303), (306), (307), in the first instance; and if m be eliminated by equation (300), these three equations will give approximate values of the masses m_2, m_3, and of m, and then m will be obtained from equation (300). But, in order to have these four quantities more accurately, their approximate values must be substituted in equations (304), (305), (308), (309), and (310), whence approximate values of

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

will be found. Again, if these approximate values of

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

be substituted in equations (303), (306), and (307), and if m be eliminated by means of equation (300), new and more accurate values of the masses and of m will be obtained. If with the last values of the masses and of m the same process be repeated, the unknown quantities will be determined with still more precision. This process must be continued till two consecutive values of each unknown quantity are nearly the same. In this manner it is found that

$$\mathrm{m} = 1.0055974;$$
$$m = 0.173281; \qquad m_1 = 0.232355;$$
$$m_2 = 0.884972; \qquad m_3 = 0.426591;$$
$$h = 0.00206221 h_3; \qquad l = 0.0207938 l_/;$$
$$h_1 = 0.0173350 h_3; \qquad l_2 = -0.0342530 l_/;$$
$$h_2 = 0.0816578 h_3; \qquad l_3 = -0.000931164 l_/.$$

893. m determines the compression of Jupiter's spheroid, for

$$r - \tfrac{1}{2} f = \mathrm{m} \cdot 0.0217794,$$

whence

$$r - \tfrac{1}{2} f = 0.0219012.$$

if t be the time of Jupiter's rotation, T the time of the sidereal revolution of the fourth satellite, then

$$f = \frac{T^2}{a_3^3 \cdot t^2}$$

is the ratio of the centrifugal force to gravity at Jupiter's equator. But

$$a_3 = 25.4359, \; T = 16.689019 \text{ days;}$$

and, according to the observations of Cassini[5] $t = 0.413889$ of a day, hence

$$f = 0.0987990, \text{ and } r = 0.0713008.$$

As the equatorial radius of Jupiter's spheroid has been taken for unity, half his polar axis will be

$$1 - r = 0.9286992.$$

The ratio of the axis of the pole to that of his equator has often been measured: the mean of these is 0.929, which differs but little from the preceding value; but on account of the great influence of the matter at Jupiter's equator on the motions of the nodes and apsides of the orbits of the satellites, this ratio is determined with more precision by observation of the eclipses than by direct measurement, however accurate.

The agreement of theory with observation in the compression of Jupiter shows that his gravitation is composed of the gravitation of all his particles, since the variation in his attractive force, arising from his observed compression, exactly represents the motions of the nodes and apsides of his satellites.

894. If the preceding values of the masses of the satellites be divided by 10,000, the ratios of these bodies to that of Jupiter, taken as the unit, are

1^{st}	0.0000173281
2^{nd}	0.0000232355
3^{rd}	0.0000884972
4^{th}	0.0000426591.

895. Assuming the values of the masses of the earth and Jupiter in article 606, the mass of the third satellite will be 0.027337 of that of the earth, taken as a unit. But it was shown that the mass of the moon is

$$\frac{1}{75} = 0.013333, \text{ \&c.}$$

of that of the earth. Thus the mass of the third satellite is more than twice as great as that of the moon, to which the mass of the fourth is nearly equal.

896. In the system of quantities,[6]

$$\begin{aligned} g_3 &= 2578''.82 \\ h &= 0.00206221 h_3 = x^{(3)} h_3 \\ h_1 &= 0.0173350 h_3 = x_1^{(3)} h_3 \\ h_2 &= 0.0816578 h_3 = x_2^{(3)} h_3 \end{aligned}$$

h_3 may be regarded as the true eccentricity of the orbit of the fourth satellite, arising from the elliptical form of the orbit, and given by observation. And the values of $h,\ h_1,\ h_2,$ are those parts of the eccentricities of the other three orbits, which arise from the indirect action of the matter at Jupiter's equator; for the attraction of that matter, by altering the position of the apsides of the fourth satellite, changes the relative position of the four orbits, and consequently alters the mutual attraction of the satellites, and is the cause of the changes in the form of the orbits expressed by the preceding values of $h,\ h_1,\ h_2$. This is the reason why these quantities depend on the annual and sidereal motion of the apsides of the fourth satellite.

897. A similar system exists for each root of g, arising from the same cause, and depending on the annual and sidereal motions of the apsides of the other three satellites. These are readily obtained from the general equations (271), which become, when the values of the masses and of the quantities in equations (301) are substituted,

$$\begin{aligned} 0 = &+\left\{ g - 185091''.3 - \frac{16613''.78}{\left(1+\dfrac{g}{972421''}\right)^2} \right\} h + \left\{ 2222''.1 - \frac{8220''.4}{\left(1+\dfrac{g}{972421''}\right)^2} \right\} h_1 \\ &+ \left\{ 270''.1 + \frac{5212''.2}{\left(1+\dfrac{g}{972421''}\right)^2} \right\} h_2 + 29''.5 h_3; \end{aligned} \qquad \textbf{(311)}$$

$$0 = +\left\{ 1313''.7 - \frac{5668''.5}{\left(1+\dfrac{g}{972421''}\right)^2} \right\} h + \left\{ g - 43214'' - \frac{15936''.3}{\left(1+\dfrac{g}{972421''}\right)^2} \right\} h_1 \qquad \textbf{(312)}$$

$$+\left\{4148''.9+\frac{6740''.6}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_2+109''.3h_3;$$

$$0=+\left\{89''.5+\frac{752''.6}{\left(1+\frac{g}{972421''}\right)^2}\right\}h+\left\{862''.5+\frac{1413''.5}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_1 \qquad \textbf{(313)}$$

$$+\left\{g-9227''.1-\frac{616''.4}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_2+552''.2h_3;$$

$$0=5''.7h+35''.53h_1+863''.74h_2+\left(g-2650''.1\right)h_3. \qquad \textbf{(314)}$$

898. As the motion of the apsides of the orbits of the satellites is almost entirely owing to the compression of Jupiter, in the first approximation the coefficient of h_2 may be made zero in equation (311); whence

$$g=9227''.1+\frac{616''.4}{\left(1+\frac{g}{972421''}\right)^2},$$

or, omitting g in the divisor,

$$g=9843''.5=10000'' \text{ nearly;}$$

hence, if $10000''$ be put for g in equations (311), (312), (314), they will give values of

$$\frac{h}{h_2},\ \frac{h_1}{h_2},\ \frac{h_3}{h_2};$$

and, by the substitution of these in equation (311), a still more approximate value of g will be found. This process must be continued till two consecutive values of g are nearly the same. In this manner it may be found that

$$g_3=9399''.17$$
$$h=0.0238111h_2=x_1^{(2)}h_2$$

$$h_1 = 0.2152920 h_2 = x_2^{(2)} h_2$$
$$h_3 = 0.1291564 h_2 = x_3^{(2)} h_2$$

h_2 may be regarded as the true eccentricity of the orbit of the third satellite, and h, h_1, h_3, are those parts of the eccentricities of the other three orbits, arising from the action of Jupiter's equator on the apsides of the third, and depending on $g_2 = 9399''.17$, their annual and sidereal motion.

899. Again, if h and h_1 be made zero in equations (311) and (312), and g omitted in the divisor, then will

$$g = 35114''.7, \qquad g_1 = 59152''.3,$$

and by the same method it will be found that

$$\begin{array}{ll} g = 196665'', & g_/ = 0.57718'' \\ h_1 = +0.0185238 h = x_1 h; & h_1 = -0.0375392 h_1 = x_1^{(1)} h_1 \\ h_2 = -0.0034337 h = x_2 h; & h_2 = -0.0436686 h_1 = x_2^{(1)} h_1 \\ h_3 = -0.00001735 h = x_3 h; & h_3 = +0.00004357 h_1 = x_3^{(1)} h_1 \end{array}$$

In these h and h_1 are the real eccentricities of the orbits of the first and second satellites, and the other values, h, h_1, h_2, h_3, &c., arise from the action of the other satellites corresponding to the roots g and $g_/$.

900. With regard to the inclinations of the orbits and the longitudes of the nodes, it appears, from article 892, that the system of inclinations for the root $p_/$ is

$$p_/ = 43374''.01$$
$$l = 0.0207938 l_/ = z_1^{(1)} l_/$$
$$l_2 = -0.0342530 l_/ = z_2^{(1)} l_/$$
$$l_3 = -0.00093116 l_/ = z_3^{(1)} l_/$$

$l_/$ is the real inclination of the orbit of the second satellite on its fixed plane, passing between the equator and orbit of Jupiter; and l, l_2, l_3, are those parts of the inclination of the other three orbits depending on the root $p_/$, and arising principally from the action of Jupiter's equator; for the attraction of that protuberant matter, by changing the place of the nodes of the second satellite, alters the relative position of the orbits, which changes the mutual attraction of the bodies, and produces the variations in the inclinations expressed by l, l_2, l_3; and it is for this reason that these quantities depend on the annual and sidereal motion of the nodes of the second satellite.

901. A similar system depends on each root of p, that is, on the annual and sidereal motions of the nodes of the orbits of the other three satellites. These are obtained from equations (307), &c.; for when the values of the masses and of m are substituted, they become

$$\begin{aligned}
0 &= (p - 185091'')\, l + 2998''.23 l_1 + 1492''.5 l_2 + 106''.03 l_3 \\
0 &= 1772''.6 l + (p - 43214'')\, l_1 + 5610''.4 l_2 + 249''.4 l_3 \\
0 &= 183''.44 l + 1166''.3 l_1 + (p - 9227''.2)\, l_2 + 813''.7 l_3 \\
0 &= 20''.4 l + 81''.09 l_1 + 1272''.8 l_2 + (p - 2650'')\, l_3 .
\end{aligned} \qquad \textbf{(315)}$$

902. The first approximate value of p is found by making the coefficient of l zero in the first of equations (315); whence $p = 185091''$; and if this value of p be put in the three last of these equations divided by l, values of $\frac{l_1}{l}, \frac{l_2}{l}, \frac{l_3}{l}$, will be found; and when these last quantities are put in the first of equations (315), a new and more correct value of p will be found: by repeating the process till two consecutive values of p nearly coincide, it will be found that

$$\begin{aligned}
p &= 185130''.14 \\
l_1 &= -0.0124527 l = z_1 l \\
l_2 &= -0.0009597 l = z_2 l \\
l_3 &= -0.0000995 l = z_3 l
\end{aligned}$$

l is the inclination of the first satellite on its fixed plane, arising chiefly from the attraction of Jupiter's equator, and given by observation; and l_1, l_2, l_3, are the parts of the inclination of the other three orbits depending on p, the annual and sidereal motion of the nodes of the first satellite.

903. The third and fourth roots of p will be obtained by making the coefficients of l_2 and l_3 respectively zero in the third and fourth of the preceding equations; and, by the same method of approximation, it will be found that

$$\begin{aligned}
p_2 &= 9193''.56, & p_3 &= 2489''.2 \\
l &= 0.0111626 l_2 = z_1^{(2)} l_2, & l &= 0.0019856 l_3 = z_1^{(3)} l_3 \\
l_1 &= 0.164053 l_2 = z_2^{(2)} l_2, & l_1 &= 0.0234108 l_3 = z_2^{(3)} l_3 \\
l_3 &= -0.196565 l_2 = z_3^{(2)} l_2, & l_2 &= 0.1248622 l_3 = z_3^{(3)} l_3,
\end{aligned}$$

where l_2 and l_3 are the real inclinations of the third and fourth satellites on their fixed planes, given by observation.

904. It now remains to compute the quantities depending on the displacement of Jupiter's equator and orbit, namely, the four values of l, $q' = \grave{L} + bt$, and $y' = \grave{p}t - \frac{at}{\grave{L}}$. The first are found by the substitution of the numerical values of the masses and of $r = \frac{1}{2}f$, in equations (285). Whence

$$l = 0.00057879$$
$$l_1 = 0.00585888$$
$$l_2 = 0.02708801$$
$$l_3 = 0.13235804.$$

Again,

$$\grave{p} = \frac{3}{4i}\left(\frac{2C - A - B}{C}\right)\left\{M^2 + mn^2 l + m_1 n_1^2 l_1 + m_2 n_2^2 l_2 + m_3 n_3^2 l_3\right\}.$$

As *A*, *B*, *C*, are the moments of inertia of Jupiter's spheroid, assumed to be elliptical, the theory of spheroids gives

$$\frac{2C - A - B}{C} = 0.14735;$$

and by observation, it is known that Jupiter's rotation is performed in 0. 41377 of a day; and that his sidereal revolution is 4332.6 days; therefore

$$\frac{M}{i} = \frac{0.41377}{4332.6};$$

then, by the substitution of the numerical values of the other quantities, all of which are given, it will appear that

$$\grave{p} = 3''.2007.$$

By observation, the inclination of Jupiter's equator on his orbit was, in 1750, $\grave{L} = 3^\circ.09996$, and as

$$a = \frac{dp}{dt}, \; b = \frac{dq}{dt}$$

are given by the theory of Jupiter at that epoch,

$$\frac{a}{\grave{L}} = 2''.93314, \; b = 0''.02279;$$

whence

$$q' = 3^\circ.09996 + 0''.02279\,t; \quad y' = 0''.2676,$$

which is nearly the annual precession of Jupiter's equinoxes on his orbit. $-y'$ expresses the longitude of the descending node of Jupiter's equator on his orbit, $180^\circ - y' = \Pi$ will be the longitude of his ascending node; consequently

$$\sin(v + y') = \sin(v - \Pi).$$

By observation, it is known that, in the beginning of 1750,

$$\Pi = 313^\circ.7592;$$

whence

$$y' = 46^\circ.241 + 0''.2676\,t\,;$$

and, with the preceding value of q', it will be found that

$$(1 - l\ )q' = 3^\circ.0899$$
$$(1 - l_1)q' = 3^\circ.0736$$
$$(1 - l_2)q' = 3^\circ.0079$$
$$(1 - l_3)q' = 2^\circ.6825.$$

905. It appears, from observation, that the two first satellites move in circular orbits, and that the first moves sensibly on its fixed plane, from the powerful attraction of Jupiter's equator; consequently *h* and $h_/$, corresponding to the roots *g* and $g_/$, are zero, as well as the inclination *l*, depending on the root *p*. Hence the systems of quantities in articles 899 and 902 are zero; and as, by observation, the real equations of the centre of the third and fourth satellite are

$$2h_2 = 245''.14, \quad 2h_2 = 553''.73, \quad 2h_3 = 3002''.04;$$

and the real inclinations of the second, third, and fourth on their fixed planes, are

$$l_1 = -1669''.31, \quad l_2 = -739''.98, \quad l_3 = -897''.998.$$

By the substitution of these quantities in the different systems,

$$x_1^{(2)}h_2,\ x_1^{(3)}h_3,\ \text{\&c. \&c.}$$

it will be found that the equations in articles 835 and 878,

$$
\begin{aligned}
dv &= +\ \ 13''.18\sin\left(nt+\in -g_2t-\Gamma_2\right) \\
&\quad -\ \ 6''.19\sin\left(nt+\in -g_3t-\Gamma_3\right) \\
dv_1 &= +\ 119''.22\sin\left(n_1t+\in_1 -g_2t-\Gamma_2\right) \\
&\quad -\ 52''.04\sin\left(n_1t+\in_1 -g_3t-\Gamma_3\right) \\
dv_2 &= -\ 552''.02\sin\left(n_2t+\in_2 -g_2t-\Gamma_2\right) \\
&\quad -\ 244''.38\sin\left(n_2t+\in_2 -g_3t-\Gamma_3\right) \\
dv_3 &= -3002''.04\sin\left(n_3t+\in_3 -g_3t-\Gamma_3\right) \\
&\quad -\ 71''.52\sin\left(n_3t+\in_3 -g_2t-\Gamma_2\right).
\end{aligned}
\qquad \textbf{(316)}
$$

$$
\begin{aligned}
s &= +3^\circ.0899\sin\left(v+46^\circ.241-49''.8t\right) \\
&\quad -34''.03\sin\left(v+p_1t+\Lambda_1\right) \\
&\quad +\ 8''.26\sin\left(v+p_2t+\Lambda_2\right) \\
s_1 &= +3^\circ.0736\sin\left(v_1+46^\circ.241-49''.8t\right) \\
&\quad -1669''.3\sin\left(v_1+p_2t+\Lambda_2\right) \\
&\quad +\ 121''.4\sin\left(v_1+p_1t+\Lambda_1\right) \\
&\quad +\ 21''.02\sin\left(v_1+p_3t+\Lambda_3\right) \\
s_2 &= +3^\circ.0079\sin\left(v_2+46^\circ.241-49''.8t\right) \\
&\quad -739''.98\sin\left(v_2+p_2t+\Lambda_2\right) \\
&\quad +112''.13\sin\left(v_2+p_3t+\Lambda_3\right) \\
&\quad +\ 57''.18\sin\left(v_2+p_1t+\Lambda_1\right) \\
s_3 &= +2^\circ.6825\sin\left(v_3+46^\circ.241-49''.8t\right) \\
&\quad -897''.998\sin\left(v_3+p_3t+\Lambda_3\right) \\
&\quad +145''.45\sin\left(v_3+p_3t+\Lambda_3\right) \\
&\quad +\ 1''.6\sin\left(v_3+p_1t+\Lambda_1\right).
\end{aligned}
\qquad \textbf{(317)}
$$

906. The following data are requisite for the complete determination of the motions of the satellites, all of them being estimated from the vernal equinox of the earth; the epoch being the instant of midnight, December 31st, 1749, mean time at Paris.

The secular mean motions of the four satellites.

$$
\begin{aligned}
n &= 7432435^\circ.47 \\
n_1 &= 3702713^\circ.2215 \\
n_2 &= 1837852^\circ.112 \\
n_3 &= 787885^\circ.
\end{aligned}
$$

The longitudes of the epochs of the satellites, estimated from the vernal equinox, were

$$\begin{aligned}\in &= 15^\circ.0128\\ \in_1 &= 131^\circ.8404\\ \in_2 &= 10^\circ.26083\\ \in_3 &= 72^\circ.5513.\end{aligned}$$

Longitudes of the lower apsides of the third and fourth satellites.

$$\begin{aligned}\Gamma_2 &= 309^\circ.438603\\ \Gamma_3 &= 180^\circ.343.\end{aligned}$$

Longitudes of ascending nodes.

$$\begin{aligned}\Lambda_1 &= 273^\circ.2889\\ \Lambda_2 &= 187^\circ.4931\\ \Lambda_3 &= 74^\circ.9687.\end{aligned}$$

The values, of g_2, g_3, &c.,p ,$p_{/}$, &c. are referred to the vernal equinox of Jupiter; but in order to refer them to the vernal equinox of the earth, the precession of the equinoxes, $= 50''$, must be added to the first and subtracted from the second; and as all the quantities in question have a1ready been given, it will be found that the annual and sidereal motions of the apsides were

$$\begin{aligned}g_2 &= 2628''.9\\ g_3 &= 9449''.28.\end{aligned}$$

The annual and sidereal motions of the nodes were

$$\begin{aligned}p_1 &= 43324''.01\\ p_2 &= 9143''.56\\ p_3 &= 2439''.08.\end{aligned}$$

Also the annual and sidereal motion of Jupiter's equinox, with regard to the vernal equinox of the earth, is

$$49''.8.$$

The longitude of Jupiter's equinox at the epoch was $46^\circ.25$, consequently

$$y' = 46^{\circ}.25 + t\,.\,49''.8,$$

and the eccentricity of Jupiter's orbit at the epoch was

$$\bar{e} = 19831''.47.$$

In order to abridge $g_2t + \Gamma_2$, $g_3t + \Gamma_3$, $pt + \Lambda$, &c., will be represented by v_2, v_3, Ω, Ω_1, Ω_2, Ω_3.

Theory of the First Satellite

Longitude

907. Since h and $h_/$ are zero, equations (302) give only the two following values of Q;

$$Q = 0.208780 \cdot h_2 = 57''.8$$
$$Q = 0.016482 \,.\, h_3 = 24''.7;$$

consequently equation (268) becomes

$$\begin{aligned} d\,v = &-57''.8\sin\left(nt - 2n_/t + \in - 2\in_/ + v_2\right) \\ &- 24''.7\sin\left(nt - 2n_/t + \in - 2\in_/ + v_3\right) \end{aligned}$$

If equation (296) and the first of equations (316) be added to this, observing that

$$2nt + 2\in - 2n_2t - 2\in_2 = 180^{\circ} + 3nt + 3\in - 3n_1t - 3\in_1,$$

it will be found that the true longitude of the first satellite in its eclipses, is

$$\begin{aligned} v = nt + \in \; &+ \; 13''.18\sin\left(nt + \in - v_2\right) \\ &+ \; 6''.19\sin\left(nt + \in - v_3\right) \\ &- \; 14''.11\sin\left(nt - n_/t + \in - \in_/\right) \\ &- \; 6''.29\sin\tfrac{3}{2}\left(nt - n_/t + \in - \in_/\right) \\ &+ 1636''.39\sin 2\left(nt - n_/t + \in - \in_/\right) \\ &+ \; 1''.22\sin 4\left(nt - n_/t + \in - \in_/\right) \\ &+ \; 0''.512\sin 5\left(nt - n_/t + \in - \in_/\right) \\ &- \; 57''.8\sin\left(nt - 2n_/t + \in - 2\in_/ + v_2\right) \\ &- \; 24''.7\sin\left(nt - 2n_/t + \in - 2\in_/ + v_3\right) \end{aligned} \qquad \textbf{(318)}$$

for in the eclipses of the satellites by Jupiter, or of Jupiter by the satellites, the longitudes of both bodies are the same; the Earth, Jupiter, and the satellites being then in the same straight line, consequently

$$Mt + E = nt + \in, \quad U = v,$$

consequently the term depending on the argument $2(nt - Mt + \in - E)$ vanishes.

Latitude

908. By article 880 the action of the sun occasions the inequality

$$s = -\frac{3M}{8n}(L' - l)\sin(v - 2U - pt - \Lambda)$$

but in the eclipses $U = v$, therefore

$$s = +\frac{3M}{8n}(L' - l)\sin(v + pt + \Lambda);$$

and as

$$l - L' = (1 - l)(L - L'),$$

and that

$$(1 - l)(L - L')\sin(v + pt + \Lambda)$$

is the latitude of the first satellite above its fixed plane, which was shown to be

$$3^\circ.0899\sin(v + 46^\circ.241 - 49''.8t),$$

therefore the preceding inequality is

$$-s = 1''.7\sin(v + 46^\circ.241 - 49''.8t).$$

When this quantity, which arises from the action of the sun, is added to the first of equations (310), it gives

$$\begin{aligned} s = &+3^\circ.0894\sin(v + 46^\circ.241 - 49''.8t) \\ &-34''.03\sin(v + \Omega_1) \\ &-\ \ 8''.26\sin(v + \Omega_2) \end{aligned}$$

for the latitude of the first satellite in its eclipses.

The inclination of the fixed plane on the equator of Jupiter is $6''.48$, which is insensible; and as the orbit has no perceptible inclination on the fixed plane, the first satellite moves nearly in a circular orbit in the plane of Jupiter's equator.

Theory of the Second Satellite

909. Because h and $h_{/}$ are insensible, equations (295) give

$$Q_{/} = -0.662615 h_2 \quad Q_{/} = -0.055035 h_3;$$

therefore equation (260) becomes

$$\begin{aligned} d v_{/} &= +183''.46 \sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + \nu_2\right) \\ &\quad - 82''.6 \sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + \nu_3\right). \end{aligned}$$

Again, equations

$$d v_{/} = \frac{5}{16} \cdot \frac{n_{/}^2}{\left(n - n_{/} - N_{/}\right)^2} \left\{mG - m_2 F'\right\}^2 \sin 2\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right)$$

$$d v_{/} = -\frac{6M}{n} \left\{1 - \frac{9 a_{/} m n^2 K}{8 a m_{/} b\left(M^2 - K n^2\right)}\right\} H \sin\left(Mt + E - \Pi\right)$$

in articles 766 and 752, have a sensible effect on the motions of the second satellite, and in consequence of

$$nt + \epsilon = 180^\circ - 2n_2 t + 3n_1 t - 2\epsilon_2 + 3\epsilon_1,$$

they become, by the substitution of the numerical values of the quantities,

$$\begin{aligned} d v_{/} &= +22''.61 \sin 4\left(n_1 t - n_2 t + \epsilon_1 - \epsilon_2\right) \\ &\quad -36''.07 \sin\left(Mt + E - \Pi\right). \end{aligned}$$

If to these the second of equations (309) be added, together with equation (290), it will be found, in consequence of the relation,

$$nt - n_{/}t + \epsilon - \epsilon_{/} = 180^\circ + 2n_1 t - 2n_2 t + 2\epsilon_1 - 2\epsilon_2,$$

that the true longitude of the second satellite is, in its eclipses,

$$\begin{aligned} d v_{/} &= n_{/} + \epsilon_{/} + 119''.22 \sin\left(n_{/}t + \epsilon_1 - \nu_2\right) \\ &\quad + 52''.04 \sin\left(n_1 t + \epsilon_1 - \nu_3\right) \end{aligned}$$

$$
\begin{aligned}
&- \quad 52''.91\sin\left(n_1t - n_2t + \in_1 - \in_2\right)\\
&+3862''.3\sin 2\left(n_1t - n_2t + \in_1 - \in_2\right)\\
&+ \quad 19''.75\sin 3\left(n_1t - n_2t + \in_2 - \in_2\right)\\
&+ \quad 24''.18\sin 4\left(n_1t - n_2t + \in_1 - \in_2\right)\\
&+ \quad 1''.51\sin 5\left(n_1t - n_2t + \in_1 - \in_2\right)\\
&+ \quad 1''.19\sin 6\left(n_1t - n_2t + \in_1 - \in_2\right)\\
&- \quad 1''.71\sin\left(n_1t - n_3t + \in_1 - \in_3\right)\\
&+ \quad 1''.5\sin 2\left(n_1t - n_3t + \in_1 - \in_3\right)\\
&+ 183''.46\sin\left(nt - 2n_1t + \in - 2\in_1 + \nu_2\right)\\
&+ \quad 82''.6\sin\left(n_1t - 2n_2t + \in_1 - 2\in_2 + \nu_3\right)\\
&- \quad 36''.07\sin\left(Mt + E - \Pi\right),
\end{aligned}
\qquad \textbf{(319)}
$$

for the last term of equation (290) vanishes.

The Latitude

910. The equation (284),

$$s_{/} = -\frac{3M}{8n_{/}}\left\{L' - l\right\}\sin\left(v_{/} - 2U - pt - \Lambda\right)$$

has a different value for each root of p, including $\grave{p}$ the root, that depends on the displacement of Jupiter's orbit and equator; but because

$$v_{/} = U, \quad \left(l_{/} - L'\right) = \left(1 - l_{/}\right)\left(L - L'\right),$$

and that

$$\left(1 - l_{/}\right)\left(L - L'\right)\sin\left(v_{/} + pt + \Lambda\right)$$

is the latitude of the second satellite above its fixed plane, which is

$$3^{\circ}.0736\sin\left(v_{/} + 46^{\circ}241 - 49''.8t\right)$$

the equation in question becomes

$$s_{/} = 3''.4\sin\left(v_{/} + 46^{\circ}241 - 49''.8t\right).$$

The only remaining root of p that gives the preceding equation a sensible value in the theory of this satellite is $p_{/} = 43324''.9$; and by the substitution of the corresponding values

$$s_{/} = 0''.512\sin\left(v_{/} + \Omega_{/}\right).$$

In consequence of these two inequalities the second of equations (310) becomes

$$\begin{aligned} s_{/} = &+3''.07262\sin\left(v_{/} + 46^{\circ}241 - 49''.8t\right) \\ &-1669''.3\sin\left(v_{/} + \Omega_{2}\right) \\ &-\ 121''.4\sin\left(v_{/} + \Omega_{1}\right) \\ &-\ 21''.04\sin\left(v_{/} + \Omega_{3}\right). \end{aligned} \qquad \textbf{(320)}$$

The inclination of the fixed plane on the equator of Jupiter is $63''.124$. The orbit of the satellite revolves on this plane, to which it is inclined at an angle of $27'48''.3$, its nodes completing a revolution in $29^{\text{yrs}}.914$.

Theory of the Third Satellite

911. The inequalities represented by

$$d\,v_{2} = -Q_{2}\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + gt + T\right)$$

have a very sensible influence on the motions of the third satellite, because observation proves that body to have two distinct equations of the centre, one depending on the lower apsis of the orbit of the second satellite, and the other on that of the fourth. Consequently h_1 and h_2 in the coefficient

$$Q_{2} = -m_{1}\frac{\left(3.248934h_{1} - 1.188133h_{2}\right)}{\left(1 + \dfrac{g}{972421''}\right)^{2}}$$

have respectively two values, namely,

$$h_{1} = 0.2152920h_{2}, \text{ and } h_{2} = -276''.865;$$

corresponding to g_2 and Γ_2, also

$$h_{1} = 0.0173350h_{3}, \text{ and } h_{2} = 0.0816578h_{3},$$

corresponding to g_3 and Γ_3; therefore the preceding inequality, in consequence of the relations among the mean longitudes of the three first satellites, gives

$$\begin{aligned} d v_2 = & -30''.84 \sin\left(n_1 t - 2n_2 t + \in_1 - 2\in_2 + \varpi_2\right) \\ & + 14''.12 \sin\left(n_1 t - 2n_2 t + \in_1 - 2\in_2 + \varpi_3\right) \end{aligned}$$

By articles 766 and 747 the action of the sun occasions the inequalities

$$\begin{aligned} d v_2 = & -\frac{12M}{n_2}\left\{1 + \frac{3a_2 mn^2 . K}{32am_2 . b\left(M^2 - Kn^2\right)}\right\}\bar{e}\sin\left(Mt + E - \Pi\right) \\ & - \frac{15Mh_2}{4n_2}\sin\left(n_2 t - 2Mt + \in - 2E + gt + T\right) \end{aligned}$$

In consequence of the two values of h_2, and because

$$2Mt + 2E = 2n_2 t + 2\in_2, \text{ in the eclipses}$$

these give

$$\begin{aligned} d v_2 = & + \ 1''.71 \sin\left(n_2 t + \in_2 - \varpi_2\right) \\ & + \ 0''.76 \sin\left(n_2 t + \in_2 - \varpi_3\right) \\ & - 47''.76 \sin\left(Mt + E - \Pi\right). \end{aligned}$$

Adding the preceding inequalities to those in (291), and to the third of (309), it will be found that the longitude of the third satellite, in its eclipses, is

$$\begin{aligned} v_2 = n_2 t + \in_2 & + 552''.031 \sin\left(n_2 t + \in_2 - \varpi_2\right) \\ & + 244''.38 \sin\left(n_2 t + \in_2 - \varpi_3\right) \\ & - 261''.86 \sin\left(n_1 t - n_2 t + \in_1 - \in_2\right) \\ & - \ 3''.84 \sin 2\left(n_1 t - n_2 t + \in_1 - \in_2\right) \\ & - \ 2''.13 \sin 3\left(n_1 t - n_2 t + \in_1 - \in_2\right) \\ & - \ 14''.65 \sin\left(n_2 t - n_3 t + \in_2 - \in_3\right) \\ & + \ 50''.06 \sin 2\left(n_2 t - n_3 t + \in_2 - \in_3\right) \\ & + \ 3''.52 \sin 3\left(n_2 t - n_3 t + \in_2 - \in_3\right) \\ & + \ 0''.82 \sin 4\left(n_2 t - n_3 t + \in_2 - \in_3\right) \\ & + \ 30''.84 \sin\left(n_1 t - 2n_2 t + \in_1 - 2\in_2 + \varpi_2\right) \\ & + \ 14''.12 \sin\left(n_1 t - 2n_2 t + \in_1 - 2\in_2 + \varpi_3\right) \\ & - \ 47''.76 \sin\left(Mt + E - \Pi\right). \end{aligned} \qquad \textbf{(321)}$$

912. The double equation of the centre, occasions some peculiarities in the motion of the third satellite. By a comparison of

$$\begin{aligned} v_2 &= 9449''.28\,t + 309^\circ.438603 \\ v_2 &= 2628''.9\,t + 180^\circ.343, \end{aligned}$$

it appears that the lower apsides of the third and fourth satellites coincided in 1682, and then the coefficient of the equation of the centre was equal to the sum of the coefficients of the two partial equations. In 1777 the lower apsis of the third satellite was 180° before that of the fourth, and the coefficient of the equation of the centre was equal to the difference of the coefficients of the partial equations; results that were confirmed by observation.

Latitude

913. The only part of the equation

$$s_2 = -\frac{3M}{2n_2}\left(L' - l_2\right)\sin\left(v_2 - 2U - pt - \Lambda\right)$$

that is sensible in the motions of the third satellite is that relating to the equator of Jupiter, whence it is easy to see that

$$s_2 = -6''.7068\sin\left(v_2 + 46^\circ.241 - 49''.8t\right);$$

the same expression with regard to the third satellite, gives

$$0''.46\sin\left(v_2 + \Omega_2\right),$$

the first subtracted from the third of equations (310), gives the latitude of the third satellite equal to

$$\begin{aligned} s_2 = &+3^\circ.0061\sin\left(v_2 + 46^\circ.241 - 49''.8t\right) \\ &-739''.53\sin\left(v_2 + \Omega_2\right) \\ &-112''.13\sin\left(v_2 + \Omega_3\right) \\ &+\ 57''.18\sin\left(v_2 + \Omega_1\right) \end{aligned} \qquad \textbf{(322)}$$

in its eclipses.

The inclination of the fixed plane of the third satellite on the equator of Jupiter is $301''.49 = l_2 q_1$. Its orbit revolves on this plane, to which it is inclined at an angle of $12'\ 20''$, the nodes accomplishing their retrograde revolution in $141^{yrs}.739$.

Theory of the Fourth Satellite

914. By article 746 the action of the sun occasions the inequalities

$$d v_3 = + \frac{15}{4} \cdot \frac{Mh_3}{n_3} \cdot \left(n_3 t + \epsilon_3 + v_3 - 2Mt - 2E \right)$$
$$- \frac{3M}{n_3} \cdot \bar{e} \cdot \sin\left(Mt + E - \Pi \right),$$

and the secular variation in the inclination of the equator and orbit of Jupiter, by article 792, occasions the inequality

$$d v_3 = - \frac{\left\{ 4\left(1 - l_3\right) \boxed{3} - \frac{1}{2}\left(1 - l_3\right) p_3 + 6\left(3\right) l_3 \right\}}{p_3} \cdot q' l_3 \sin\left(p_3 t + \Lambda_3 - y' \right)$$

It is easy to see that the two first inequalities are,

$$d v_3 = + \;\; 21''.69 \sin\left(n_3 t + \epsilon_3 + v_3 - 2Mt - 2E \right)$$
$$-133''.33 \sin\left(Mt + E - \Pi \right);$$

but in the eclipses $Mt + E = n_3 t + \epsilon_3$. So

$$d v_3 = - \;\; 21''.69 \sin\left(n_3 t + \epsilon_3 - v_3 \right)$$
$$-133''.33 \sin\left(Mt + E - \Pi \right),$$

and the third inequality is[7]

$$d v_3 = -16''.04 \sin\left(28^\circ.812 + 2488''.91 t \right).$$

If these be added to equation (292), and the last of equations (309), the longitude of the fourth satellite in its eclipses is,

$$\begin{aligned} v_3 = n_3 t + \epsilon_3 & + 2980''.35 \sin\left(n_3 t + \epsilon_3 - v_3 \right) \\ & + \;\; 13''.65 \sin 2\left(n_3 t + \epsilon_3 - v_3 \right) \\ & + \;\;\;\; 0''.09 \sin 3\left(n_3 t + \epsilon_3 - v_3 \right) \\ & - \;\; 71''.28 \sin\left(n_3 t + \epsilon_3 - v_2 \right) \\ & - \;\; 10''.16 \sin\left(n_3 t - n_2 t + \epsilon_3 - \epsilon_2 \right) \\ & - \;\;\;\; 4''.58 \sin 2\left(n_3 t - n_2 t + \epsilon_3 - \epsilon_2 \right) \\ & - \;\;\;\; 0''.96 \sin 3\left(n_3 t - n_2 t + \epsilon_3 - \epsilon_2 \right) \\ & - \;\;\;\; 0''.29 \sin 4\left(n_3 t - n_2 t + \epsilon_3 - \epsilon_2 \right) \end{aligned} \qquad \textbf{(323)}$$

$$
\begin{aligned}
&- \quad 0''.11\sin 5\left(n_3 t - n_2 t + \epsilon_3 - \epsilon_2\right) \\
&- 113''.33\sin\left(Mt + E - \Pi\right) \\
&- \quad 16''.04\sin\left(2488''.91t + 28^\circ.73\right).
\end{aligned}
$$

The terms having the coefficients $13''.65$ and $0''.09$ belong to the equation of the centre, which in this satellite extends to the squares and cubes of the eccentricity.

Latitude

915. The inequality of article 789

$$s_3 = \frac{3M}{8n_3}\left(l_3 - L'\right)\sin\left(v_3 - 2U - \ pt - \Lambda\right),$$

arising from the action of the sun, has two sensible values, one arising from the displacement of Jupiter's orbit, and the other depending on the inclination of the orbit of the fourth satellite on its fixed plane. Because

$$l_3 - L' = \left(1 - l_3\right)\left(L - L'\right) = 2^\circ.6825,$$

the first of these inequalities is

$$s_3 = 13''.98\sin\left(v_3 + 46^\circ.241 - 49''.8t\right),$$

in the eclipses when $U = v_3$, and the other depending on

$$p_3 = 2439''.08$$

is in the eclipses

$$s_3 = 1''.3\sin\left(v_3 + \Omega_3\right).$$

Adding these to the last of equations (310) the latitude of the fourth satellite in its eclipses is[8]

$$
\begin{aligned}
s_3 = &+2^\circ.6786\sin\left(v_3 + 46^\circ.241 - 49''.8t\right) \\
&-896''.702\sin\left(v_3 + \Omega\right) \\
&+145''.46\sin\left(v_3 + \Omega\right) \\
&+ \quad 1''.6\sin\left(v_3 + \Omega\right).
\end{aligned}
\qquad \mathbf{(324)}
$$

916. The inclination of the fixed plane of the fourth satellite on Jupiter's equator is

$$l_3 q' = 1473''.14.$$

The orbit of the satellite revolves on that plane to which it is inclined at an angle of [9] $14' \; 58''$; its nodes accomplish a revolution in 531 years.

917. The preceding expression for the latitude explains a singular phenomenon observed in the motion of the fourth satellite. The inclination of its orbit on the orbit of Jupiter appeared to be constant, and equal to $2^\circ.43$ from the year 1680 to 1760; during that time the nodes had a direct motion of about $4'.32$ annually. From 1760 the inclination has increased. The latitude may be put under the form

$$A \sin v_3 - B \cos v_3;$$

A and *B* will be determined by making

$$v_3 = 90^\circ, \text{ and } v_3 = 180^\circ$$

successively in the expression s_3; $\frac{B}{A}$ will be the tangent of the longitude of the node and $\sqrt{A^2 + B^2}$, the inclination of the orbit. If then t be successively made equal to -70; -30; and 10 which corresponds to the years 1680, 1720, and 1760, estimated from the epoch of 1750, the result will be[10]

	Inclination	Longitude Ω
1680	$2^\circ.4764$	$311^\circ.4172$
1720	$2^\circ.4489$	$313^\circ.3067$
1760	$2^\circ.4411$	$317^\circ.0914$

If the inclination be represented by

$$2^\circ.4764 + Nt + Pt^2$$

t being the number of years elapsed since 1680. Comparing this formula with the preceding inclination

$$N = -0^\circ.0009315 \quad P = 0^\circ.000061313.$$

The minimum of the formula corresponds to t = 75.953, or to the year 1756. The mean of the three preceding values is $2^\circ.4555$, and the mean annual motion of the node from 1680 to 1760 is $4'.255$. These results are conformable to observation during this interval, but from 1760 the inclination has varied sensibly. The preceding value of s_3 gives an inclination of $2^\circ.5791$ in 1800, and the longitude of the node equal to $320^\circ.2935$; and as observation confirms these results, it must be concluded, that the inclination is a variable quantity, but the law of the variation could hardly have been determined independently of theory.

Notes

[1] This chapter is numbered VIII in the 1[st] edition.

[2] The subscript on the 2[nd] equation is omitted in the 1[st] edition; it *reads* $Q = -m_1$ &c. The third equation *reads* $Q_3 - m_1$ &c.

[3] The element $594''.41m$ in equation (305) *reads* $594''.41m'$ in the 1[st] edition. The element $89''.7$ in equation (306) *reads* $89'.7$ in the 1[st] edition. The punctuation after equation (307) is altered from a semicolon to a period.

[4] This *reads* $g_3 = 2578''.82_2$ in the 1[st] edition.

[5] See note 53, *Bk. II, Chap. XIV*.

[6] The right hand sides of the 2[nd], 3[rd], and 4[th] equations all *read* $x_1^{(3)} h_3$ in the 1[st] edition.

[7] The coefficient $-16''.04$ *reads* -16.04 in the 1[st] edition.

[8] The coefficient of the fourth term *reads* $1'6$ in the 1[st] edition.

[9] This *reads* $14'.58'$ in the 1[st] edition.

[10] The year "1720" in the following table *reads* "1620" in the 1[st] edition.