BOOK IV

CHAPTER II[1]

PERTURBATIONS OF THE SATELLITES IN LATITUDE

859. THE perturbations in latitude are found with most facility from

$$0=\frac{d^2s}{dv^2}\left\{1-\frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\}-\frac{1}{h^2u^2}\cdot\frac{ds}{dv}\cdot\left(\frac{dR}{dt}\right)+\frac{s}{h^2u^2}\left(\frac{dR}{du}\right)+\frac{\left(1+s^2\right)}{h^2u^2}\left(\frac{dR}{ds}\right),$$

which was employed for the moon, but in that case R was a function of u, v, and s, and the differential $\frac{dR}{ds}$ was taken in that hypothesis, which we shall represent by $\frac{dR'}{ds}$, but now R is a function of r, v, and s, hence

$$du\left(\frac{dR}{du}\right)+ds\left(\frac{d'R}{ds}\right)=dr\left(\frac{dR}{dr}\right)+ds\left(\frac{dR}{ds}\right)$$

and as

$$r=\frac{\sqrt{1+s^2}}{u},\quad du=-\frac{dr}{r^2}\sqrt{1+s^2}+\frac{sds}{r\sqrt{1+s^2}};$$

and comparing the coefficients of ds in these two equations

$$\frac{us}{1+s^2}\left(\frac{dR}{du}\right)+\frac{d'R}{ds}=\left(\frac{dR}{ds}\right),$$

so the preceding equation of latitude, when $\frac{\sqrt{1+s^2}}{r}$ is put for u, and the product of the disturbing force by $s^2\cdot\frac{ds}{dv}$ omitted, becomes

$$0=\frac{d^2s}{dv^2}+s+\frac{r^2}{h^2}\left(\frac{dR}{ds}\right)-\frac{r^2}{h^2}\cdot\frac{ds}{dv}\left(\frac{dR}{dv}\right).$$

860. The only part of the disturbing force that affects the latitude is

$$R = -\frac{m_/ r_/ r\left\{ss_/ - \frac{1}{2}s^2\cos\left(v_/ - v\right)\right\}}{\left\{r^2 - rr_/\cos\left(v_/ - v\right) + r_/^2\right\}^{\frac{3}{2}}} + \frac{3m_/}{4n^2}\left\{s_/^2 + s^2\cos\left(2v - 2u\right) - 4ss_/\cos\left(v - u\right)\right\} + \frac{\left(r - \frac{1}{2}f\right)}{a^2}\left(s - s'\right)^2.$$

If the eccentricities be omitted, $r = a$, $r_/ = a_/$, and

$$\frac{1}{\left(r^2 - 2rr_/\cos\left(v_/ - v\right) + r_/^2\right)^{\frac{3}{2}}} = \left\{a^2 - 2aa_/\cos\left(v_/ - v\right) + a_/^2\right\}^{-\frac{3}{2}} = \tfrac{1}{2}B_0 + B_1\cos\left(v_/ - v\right) + \text{\&c.}$$

as for the planets; hence

$$R = -\sum m_/ a^2 a_/\left\{ss_/ - \tfrac{1}{2}s^2\cos\left(v_/ - v\right)\right\}B_1\cos\left(v_/ - v\right) + \frac{3M^2}{4n^2}\left\{s^2 - 4sS\cos\left(v - U\right)\right\} + \frac{\left(r - \frac{1}{2}f\right)}{a^2}\left(s - s'\right)^2.$$

Whence

$$0 = +\frac{d^2s}{dt^2} + s\left\{1 + 2\frac{\left(r - \frac{1}{2}f\right)}{a^2} + \frac{3}{2}\frac{M^2}{n^2} + \tfrac{1}{2}\sum m_/ a^2 a_/ \,.\, B_1\right\}$$
$$-\frac{2\left(r - \frac{1}{2}f\right)}{a^2}s' - \frac{3M^2}{n^2}S\cos\left(U - v\right) - \sum m_/ a^2 \cdot a_/ B_1 s_/\cos\left(v_/ - v\right).$$

861. In order to integrate this equation, let

$$s = l\,.\sin\left(v + pt + \Lambda\right);\quad s_/ = l_/\sin\left(v_1 + pt + \Lambda\right)$$
$$s_2 = l_2\,.\sin\left(v_2 + pt + \Lambda\right);\quad s_3 = l_3\sin\left(v_3 + pt + \Lambda\right)$$
$$S = L'\cdot\sin\left(U + pt + \Lambda\right);\quad s' = L\sin\left(v + pt + \Lambda\right),$$

l, l_1, l_2, l_3, L' and L being the inclination of the orbits of the four satellites, of Jupiter's orbit and equator on the fixed plane, p and Λ, quantities on which the sidereal motions and longitudes of the nodes depend.

If the motion of only one satellite be considered at a time, then substituting for s, $s_/$ and S, also putting $\frac{v}{n}$ for t, and omitting p^2, the comparison of the coefficients of $\sin\left(v + pt + \Lambda\right)$ gives

$$0 = +l\left\{\frac{p}{n} - \frac{\left(r - \tfrac{1}{2}f\right)}{a^2} - \tfrac{3}{4}\frac{M^2}{n^2} - \tfrac{1}{4}\sum m_/ a^2 \,.\, a_/ B_1\right\} + \frac{\left(r - \tfrac{1}{2}f\right)}{a^2}L + \tfrac{3}{4}\frac{M^2}{n^2}\cdot L' + \tfrac{1}{4}\sum m_/ a^2 \cdot a_/ B_1 l_/. \qquad \textbf{(275)}$$

If $\dfrac{a}{a'_/} = \alpha$, and $v_/ - v = n_/ t - nt + \epsilon_/ - \epsilon = \beta$

$$\left\{1 - 2\alpha\cos\beta + \alpha^2\right\}^{-\frac{3}{2}} = a_/^3\left(\tfrac{1}{2}B_0 + B_1\cos\beta + B_2\cos 2\beta + \&\text{c.}\right),$$

which is identical with the series in article 446, and therefore the formulae for the planets give by article

$$(0.1) = \frac{m_1 n \cdot a^2 a_1 B_1}{4},$$

consequently equation (275) becomes

$$0 = l\left\{p - (0) - \boxed{0} - (0.1)\right\} + L(0) + L'\boxed{0} + (0.1)\,l_/,$$

but the action of the satellites m_2 and m_3 produce terms analogous to those produced by m_1; so the preceding equation, including the disturbing action of all the bodies, and the compression of Jupiter, is

$$0 = +l\left\{p - (0) - \boxed{0} - (0.1) - (0.2) - (0.3)\right\} + (0)L + \boxed{0}L' + (0.1)\,l_1 + (0.2)\,l_2 + (0.3)\,l_3. \qquad \textbf{(276)}$$

By the same process the corresponding equations for the other satellites are

$$\begin{aligned}0 &= +l_1\left\{p - (1) - \boxed{1} - (1.0) - (1.2) - (1.3)\right\}\\ &\quad + (1)L + \boxed{1}L' + (1.0)\,l + (1.2)\,l_2 + (1.3)\,l_3;\\ 0 &= +l_2\left\{p - (2) - \boxed{2} - (2.0) - (2.1) - (2.3)\right\}\\ &\quad + (2)L + \boxed{2}L' + (2.0)\,l + (2.1)\,l_1 + (2.3)\,l_3;\\ 0 &= +l_3\left\{p - (3) - \boxed{3} - (3.0) - (3.1) - (3.2)\right\}\\ &\quad + (3)L + \boxed{3}L' + (3.0)\,l + (3.1)\,l_1 + (3.2)\,l_2.\end{aligned} \qquad \textbf{(277)}$$

862. These four equations determine the coefficients of the latitude; they include the reciprocal action of the satellites, together with that of the sun, and the direct action of Jupiter considered as a spheroid, but in the hypothesis that the plane of his equator retains a permanent inclination on the fixed plane: that, however, is not the case, for as neither the sun nor the orbits of all the satellites are in the plane of Jupiter's equator, their action on the protuberant matter causes a nutation in the equator, and a precession of its equinoxes, in all respects similar to those occasioned by the action of the moon on the earth, which produce sensible inequalities in the motions of the satellites. Thus the satellites, by troubling Jupiter, indirectly disturb their own motions.

The Effect of the Nutation and Precession of Jupiter on the Motion of his Satellites

863. The reciprocal action of the bodies of the solar system renders it impossible to determine the motion of any one part independently of the rest; this creates a difficulty of arrangement, and makes it indispensable to anticipate results which can only be obtained by a complete investigation of the theory on which they depend. The nutation and precession of Jupiter's spheroid can only be known by the theory of the rotation of the planets, from whence it is found that if q and g be the inclinations of Jupiter's equator and orbit on the fixed plane, y the retrograde motion of the descending node of his equator on that plane, and estimated from the vernal equinox of Jupiter, t the longitude of the ascending node of his orbit, *it* the rotation of Jupiter, and *A*, *B*, *C*, the moments of inertia of his spheroid with regard to the principal axes of rotation, as in article 177, the precession of Jupiter's equinoxes is

$$\frac{dq}{dt}=\frac{3(2C-A-B)}{4iC}\left\{M^2 g\sin(t+y)+\sum mn^2 g'\sin(t'+y)\right\}$$

whence $M^2 g\sin(t+y)$ is the action of the sun, and $\sum mn^2 g'\sin(t'+y)$ that of the satellites. The nutation is

$$q\cdot\frac{dy}{dt}=\frac{3(2C-A-B)}{4iC}\left\{q\left(M^2+\sum mn^2\right)+M^2 g\sin(t+y)+\sum mn^2 g'\sin(t'+y)\right\}.$$

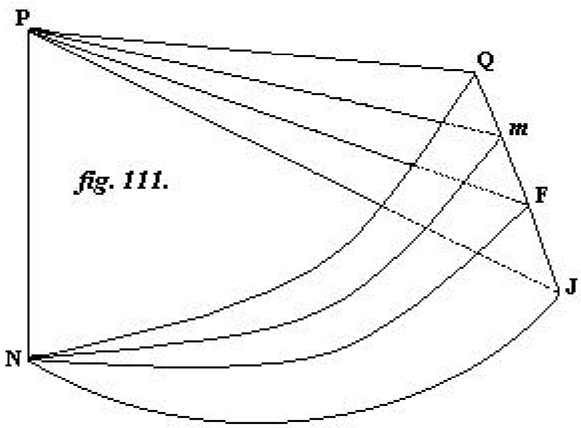
The first of these equations, multiplied by $\sin y$, added to the second multiplied by $\cos y$, gives

$$\frac{d(q\sin y)}{dt}=\frac{3(2C-A-B)}{4iC}\left\{\left(M^2+\sum mn^2\right)q\sin y+M^2 g\cos t+\sum mn^2 g'\cos t'\right\} \qquad \mathbf{(278)}$$

likewise

$$\frac{d(q\cos y)}{dt}=\frac{3(2C-A-B)}{4iC}\left\{-\left(M^2+\sum mn^2\right)q\sin y+M^2 g\sin t+\sum mn^2 g'\sin t'\right\} \qquad \mathbf{(279)}$$

864. Now, in order to have some idea of the positions of the different planes, let JN be the orbit of Jupiter, QN the plane of his equator, FN the fixed plane, and *m*N the orbit of a



satellite. Then the integrals of these equations may be found by considering that as $q = \text{QNF}$ is the inclination of Jupiter's equator on the fixed plane, $-q\sin(v+y)$ would be the latitude of a satellite if it moved on the plane of Jupiter's equator, for the latitudes are all referred to the fixed plane FN; and if they are positive on the side FJ, they must be negative on the side FQ; but by the value assumed for s, in article 861, that latitude is equal to a series of terms of the form

$$L\sin(v+pt+\Lambda),$$

hence

$$\begin{aligned} q\sin y &= -\Sigma'.L\sin(pt+\Lambda) \\ q\cos y &= -\Sigma'.L\cos(pt+\Lambda). \end{aligned} \quad \textbf{(280)}$$

865. Likewise, $g = \text{JNF}$, being the inclination of Jupiter's orbit on the fixed plane, $g\sin(U-t)$ is the latitude of the sun above the fixed plane, by article 863; but by the value assumed for *S*, in article 861, it is easy to see that

$$\begin{aligned} g\sin t &= -\Sigma'.L'\sin(pt+\Lambda) \\ g\cos t &= -\Sigma'.L'\cos(pt+\Lambda). \end{aligned} \quad \textbf{(281)}$$

In the same manner $g' = m\text{NF}$ being the inclination of the orbit of a satellite on the fixed plane, its latitude is $g'\sin(v+l_{/})$, and by article 861

$$\begin{aligned} g'\sin t' &= -\Sigma' l\,.\sin(pt+\Lambda) \\ g'\cos t' &= +\Sigma' l\cdot\cos(pt+\Lambda). \end{aligned} \quad \textbf{(282)}$$

Σ' denotes the sum of a series, but Σ is the sum of the terms relating to the different satellites.

When these quantities are put in equations (279) and (278), a comparison of the coefficients of similar sines and cosines gives

$$0 = pL + \frac{3(2C-A-B)}{4iC}\left\{M^2(L'-L) + \Sigma mn^2(l-L)\right\},$$

which equation determines the effect of the displacement of Jupiter's equator.

866. If Jupiter be an elliptical spheroid, theory gives

$$\frac{2C-A-B}{C}=\frac{2\left(r-\frac{1}{2}f\right)\int P.\bar{R}^2 d\bar{R}}{\int P\bar{R}^4\cdot d\bar{R}}.$$

As the celestial bodies decrease in density from the centre to the surface, P represents the density of a shell or layer of Jupiter's spheroid at the distance of $\bar{R}$ from his centre, the integral being between $\bar{R}=0,$ the value of the radius at the center to $\bar{R}=1,$ its value at the surface.

867. The equations[2] (277) may be put under the form

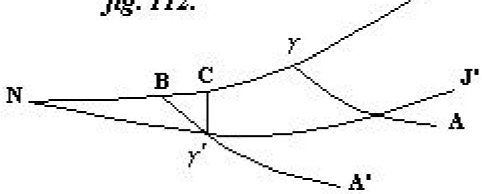
$$\begin{aligned}
0=&+\left\{p-(0)-\boxed{0}-(0.1)-(0.2)-(0.3)\right\}(L-l)\\
&+(0.1)(L-l_1)+(0.2)(L-l_2)+(0.3)(L-l_3)+\boxed{0}(L-L')-pL;\\
0=&+\left\{p-(1)-\boxed{1}-(1.0)-(1.2)-(1.3)\right\}(L-l_1)\\
&+(1.0)(L-l)+(1.2)(L-l_2)+(1.3)(L-l_3)+\boxed{1}(L-L')-pL;\\
0=&+\left\{p-(2)-\boxed{2}-(2.0)-(2.1)-(2.3)\right\}(L-l_2)\\
&+(2.0)(L-l)+(2.1)(L-l_1)+(2.3)(L-l_3)+\boxed{2}(L-L')-pL;\\
0=&+\left\{p-(3)-\boxed{3}-(3.0)-(3.1)-(3.2)\right\}(L-l_3)\\
&+(3.0)(L-l)+(3.1)(L-l_1)+(3.2)(L-l_2)+\boxed{3}(L-L')-pL;\\
0=&+pL-\frac{3(2C-A-B)}{4iC}\left\{\begin{array}{l}+M^2(L-L')+mn^2(L-l)+m_1n_1^2(L-l_1)\\+m_2n_2^2(L-l_2)+m_3n_3^2(L-l_3)\end{array}\right\};
\end{aligned}\qquad\textbf{(283)}$$

which determine the positions of the orbits of the satellites, including the effects of Jupiter's nutation and precession.

Inequalities occasioned by the Displacement of Jupiter's Orbit

868. The position of Jupiter's orbit is perpetually changing by very slow degrees with regard to the ecliptic, from the action of the planets. In consequence of this displacement the inclination of the plane of Jupiter's equator on his orbit is changed, and a corresponding change takes place in the precession of the nodes of the equator on the orbit, which may be represented

fig. 112.



by considering JN to be the orbit of Jupiter, and gA the plane of his equator at the epoch, g being the ascending node of his equator at that period. After a time the action of the planets brings the orbit into the position J'N, which increases the inclination by JNJ', and the node, which by its retrograde motion alone would come to B, is brought to g', so that the motion of the node is lessened by BC. Thus the inclination of the plane of Jupiter's equator on his orbit is affected by two totally different and unconnected causes, the one arising from the direct action of the sun and satellites on the protuberant matter at the equator producing nutation and precession, and the other from the displacement of his orbit by the action of the planets, which diminishes the precession: both disturb the motions of the satellites; but in order to determine the effects of the displacement of the orbit, it must be observed that if the inclinations of the orbits were eliminated from the equations (283) the result would be an equation in *p*, the roots of which $p,\ p_1,\ p_2,\ p_3,$ would be the annual and sidereal motions of the nodes of the satellites depending on their mutual attraction and that of the sun, but there would also be very small values of *p* of the order BC, fig. 112, depending on the displacement of the orbit and equator of Jupiter. Now if we regard the equations (283) as relative to the displacement of the orbit and equator of Jupiter alone, omitting, for the present, the mutual action of the sun and satellites, these very small values of *p* may be omitted in comparison of $(0),\ (0.1),$ &c.; and if it be assumed that

$$\begin{aligned} L-l &= l\,(L-L') \\ L-l_1 &= l_1(L-L') \\ L-l_2 &= l_2(L-L') \\ L-l_3 &= l_3(L-L'), \end{aligned} \qquad \textbf{(284)}$$

the four first of equations (283) become

$$\begin{aligned} 0 &= +\left\{(0)+\boxed{0}+(0.1)+(0.2)+(0.3)\right\} l \\ &\quad -(0.1)\,l_1-(0.2)\,l_2-(0.3)\,l_3-\boxed{0} \\ 0 &= +\left\{(1)+\boxed{1}+(1.0)+(1.2)+(1.3)\right\} l_1 \\ &\quad -(1.0)\,l-(1.2)\,l_2-(1.3)\,l_3-\boxed{1} \\ 0 &= +\left\{(2)+\boxed{2}+(2.0)+(2.1)+(2.3)\right\} l_2 \\ &\quad -(2.0)\,l-(2.1)\,l_2-(2.3)\,l_3-\boxed{2} \\ 0 &= +\left\{(3)+\boxed{3}+(3.0)+(3.1)+(3.2)\right\} l_3 \\ &\quad -(3.0)\,l-(3.1)\,l_1-(3.2)\,l_2-\boxed{3}, \end{aligned} \qquad \textbf{(285)}$$

which are relative to the displacement of the orbit and equator of Jupiter; by these, $l,\ l_1,\ l_2,\ l_3,$ may be computed, whence the relations among the inclinations will be known.

On the Constant Planes

869. The preceding equations afford the means of finding the permanent planes mentioned in article 803, for $l = m\text{NF}$ and $L' = \text{JNF},$ fig. 111, are the inclinations of the satellite and Jupiter on the fixed plane; $l - L' = m\text{NJ}$ is the inclination of the orbit of the satellite on that of Jupiter, by article 864; hence, the latitude of the satellite *m* above the orbit of Jupiter is equal to a series of terms of the form

$$(l-L')\sin(v+pt+\Lambda).$$

But the first of equations (284) gives

$$l-L'=(1-l)(L-L');$$

thus, with regard to the displacement of Jupiter's orbit and equator,

$$\Sigma'(l-L')\sin(v+pt+\Lambda)=(1-l)\Sigma'(L-L')\sin(v+pt+\Lambda).$$

Again, $L=\mathrm{QNF}$, $L_{/}=\mathrm{JNF}$ being the inclinations of Jupiter's equator and orbit on the fixed plane; $L-L'=\mathrm{QNJ}$ is the inclination of his equator on his orbit, for $L=\mathrm{QNF}$ is a negative quantity by article 864, therefore

$$\Sigma'(L-L')\sin(v+pt+\Lambda)$$

would be the latitude of the satellite *m* above the orbit of Jupiter, if it moved on the plane of his equator. But the inclination $(1-l)\times(L-L')$ is less than $L-L'=\mathrm{QNJ}$, both being referred to the plane of Jupiter's orbit; hence, $(1-l)(L-L')=l-L'<L-L'$; therefore the plane having the inclination $l-L'$, or $(1-l)(L-L')$ must come between the equator and orbit of Jupiter; and as Λ and *p*, the longitude of the node and its annual and sidereal precession, are the same in both, this plane passes through NP, the line of the nodes. But

$$L-L':(1-l)(L-L')::1:1-l::\mathrm{QNJ}:\mathrm{FNJ},$$

and the plane FN always retains the same inclination to the equator and orbit of Jupiter, since l is a constant quantity: each of the other satellites has its own permanent plane depending on l_1, l_2, l_3. It is hardly possible that these planes could have been discovered by observation alone.

870. If $q'=\mathrm{QNJ}=L-L'$ be the inclination of Jupiter's equator on his orbit, and $-y'=pt+\Lambda$ the longitude of its descending node on the orbit estimated from the vernal equinox of Jupiter, the preceding expression, with regard to that part of the latitude of *m* above the orbit of Jupiter which is relative to the displacement of his orbit and equator, is

$$(l-1)q'\sin(v+y'),$$

for $lq'=\mathrm{FNQ}$, the inclination of the constant plane FN on Jupiter's equator, therefore

$$(l-1)q' = \text{FNJ},$$

is the inclination of the same constant plane on Jupiter's orbit, and

$$(l-1)q' \cdot \sin(v+y')$$

is the latitude the satellite would have if it moved on its constant plane.

To determine the Effects of the Displacements of the Equator and Orbit of Jupiter on the quantities $q = \text{QNF}$, $q' = \text{QNJ}$, y, y', *and* Λ

871. The displacements of the equator and orbit of Jupiter affect the quantities q, y, q', y', and Λ. The general equations which determine this effect may easily be found; but if the values of these quantities be obtained in functions of the time, it will be sufficiently correct for astronomical purposes for several centuries, before or after any period that may be assumed as the epoch.

It will answer the same purpose, and facilitate the determination of these quantities, if Jupiter's orbit be assumed to coincide with the fixed plane FN; for the whole effect of its displacement will be referred to the equator, which will then vary both from nutation and the variation in the orbit of Jupiter. In this case $L' = 0$, and equations (284) give

$$l = (1-l)L; \;\; l_1 = (1-l_1)L; \;\; l_2 = (1-l_2)L; \;\; l_3 = (1-l_3)L.$$

In consequence of these, the four first of equations (283) vanish, and L remains indeterminate, and may be represented by[3] $-\grave{L}$, and the last of the same equation becomes

$$p = \frac{3(2C-A-B)}{4iC}\left\{M^2 + mn^2 l + m_1 n_1^2 l_1 + m_2 n_2^2 l_2 + m_3 n_3^2 l_3\right\},$$

which may be expressed by $\grave{p}$, and relates to the displacement of the equator of Jupiter.

872. Since JN coincides with FN, fig. 111, $-\grave{L} = \text{Q}n\text{J}$ is the inclination of the equator on the fixed orbit of Jupiter and

$$-\grave{L}\sin(v + \grave{p}t + \grave{\Lambda})$$

would be the latitude of the satellite if it were moving on the equator of Jupiter, $\grave{\Lambda}$ being an arbitrary quantity, or the longitude of the node of the equator corresponding to $\grave{p}$. But this latitude has also been expressed by $-q\sin(v+y)$. Whence[4]

$$q \sin y = `L \sin\left(`pt + `\Lambda\right),$$
$$q \cos y = `L \cos\left(`pt + `\Lambda\right), \qquad \textbf{(286)}$$

$`pt$ being the mean precession of the equinoxes of Jupiter. Again, since $q = \text{QNF}$, $g = \text{JNF}$ are the inclinations of the equator and orbit of Jupiter on the fixed plane;

$$-q \sin\left(v + y\right) - g \sin\left(v - t\right)$$

is the latitude the satellite would have above the orbit of Jupiter, if it moved on the plane of his equator, but $-q' \sin\left(v + y'\right)$ is the same; so

$$q \sin\left(v + y\right) + g \sin\left(v - t\right) = q' \sin\left(v + y'\right),$$

v being indeterminate. If it be successively made equal to $-`pt$ and to $90^\circ - `pt,$ the preceding equation gives

$$q' \sin\left(y' - pt\right) = q \sin\left(y - `pt\right) - g \sin\left(t + `pt\right)$$
$$q' \cos\left(y' - `pt\right) = q \cos\left(y - `pt\right) + g \cos\left(t + `pt\right). \qquad \textbf{(287)}$$

The sum of the squares of equations (286) gives $q = `L,$ and as by this $\sin y = \sin\left(`pt + `\Lambda\right);$ therefore $y - `pt = `\Lambda.$

In consequence of this, the first of equations (287) becomes

$$q' \sin\left(y' - `pt\right) = `L \sin `\Lambda - g \sin\left(t + `pt\right),$$

or

$$q' \sin y' \cos `pt - q_{/} \cos y_{/} \sin `pt - `L \sin `\Lambda + g \sin t \cos `pt + g \cos t \sin `pt = 0 .$$

This expression must be zero, whatever the time may be, which can only happen when $\sin `\Lambda = 0,$ for $`L = q$; consequently,

$$`\Lambda = 0,$$

and therefore

$$y = `pt .$$

873. In order to determine q' and $y',$ let the orbit of Jupiter in the beginning of 1750 be the fixed plane, let that period be the epoch, and the line of the vernal equinox of Jupiter the origin of the angles. Then at the epoch $t = 0,$ whence equations (287) become

$$q' \sin y' = q \sin y - g \sin t$$
$$q' \cos y' = q \cos y + g \cos t .$$

Now y' and y are so small, that the arc may be put for the sine, and unity for the cosine; also $g\cos t$, $g\sin t$ may be expressed by series increasing as the powers of the time for many centuries to come; therefore let

$$g\ \sin t = at \qquad g\ \cos t = bt$$

then, because

$$q = \grave{L}, \quad y = \grave{}pt, \quad \grave{\Lambda} = 0, \qquad \textbf{(288)}$$
$$q'y' = \grave{L}\,.\,\grave{}pt - at; \qquad q' = \grave{L} + bt$$

whence

$$y' = \grave{}pt - \frac{at}{\grave{L}},$$

when the square of the time is neglected.

874. Since $g\ \sin t$, $g\ \cos t$, relate to the change in the position of Jupiter's orbit from the action of the planets, they are determined by equations (137); but as Jupiter's orbit is principally disturbed by the action of Saturn and Uranus, if f, f', be the inclinations of the orbits of Saturn and Uranus on the orbit of Jupiter in the beginning of 1750, and Ω, Ω', the longitudes of the ascending nodes of the two orbits on that of Jupiter at the same epoch, estimated from the equinox of spring of Jupiter; then will

$$a = +(4.5)f\cos\Omega + (4.6)f'.\cos\Omega'$$
$$b = -(4.5)f\ \sin\Omega - (4.6)f'\cdot\sin\Omega',$$

where (4.5), (4.6), are given by equations (202).

875. It only remains to determine the effects of the displacement on $g'\sin t$, $g'\sin t'$, the inclination and node of a satellite m with regard to its fixed plane.

By equations (248)

$$l = (1 - l)\,L + l\,L'.$$

If this value of *l* be put in the equations (282) they become

$$g'\sin t' = -\Sigma'.(1 - l).L.\sin(pt + \Lambda) - \Sigma'.\ l\,L'.\sin(pt + \Lambda)$$
$$g'\cos t' = +\Sigma'.(1 - l).\,L\cdot\cos(pt + \Lambda) + \Sigma'.\,l\,L'\cdot\cos(pt + \Lambda),$$

and in consequence of equations (280) and (281)

$$g'\ \sin t' = (1 - l).q\ \sin y\ + l\ .\sin t$$
$$g'\cos\ t' = (l - 1)\cdot q\cos y\ + l\ \cdot\cos t,$$

but

$$q = \grave{}L, \quad y = \grave{}pt, \quad g \sin t = at, \quad g \cos t = bt;$$

and putting $\grave{}pt$ for the sine and unity for the cosine; with regard to the displacement of Jupiter's orbit and equator,

$$\begin{aligned} g' \sin t' &= (1 - l)\grave{}L . \grave{}pt + l . at \\ g' \cos t' &= (l - 1)\grave{}L + l \cdot bt \end{aligned} \qquad \textbf{(289)}$$

Thus the quantities relating to the displacement of the orbit and equator are completely determined.

876. With regard to the values of p, which depend on the mutual action of the satellites, L' is zero, since the action of the satellites has no sensible effect on the displacement of Jupiter's orbit. The values of L may be omitted relatively to l, l_1, l_2, l_3; and since by the last of equations (283), pL is multiplied by $\dfrac{2C - A - B}{C}$, it is of the order of the product of the ellipticity of Jupiter by the masses of the satellites; and therefore it may be omitted also, which reduces equations (283) to

$$\begin{aligned} 0 &= +l\left\{p - (0) - \boxed{0} - (0.1) - (0.2) - (0.3)\right\} + (0.1)l_1 + (0.2)l_2 + (0.3)l_3 \\ 0 &= +l_1\left\{p - (1) - \boxed{1} - (1.0) - (1.2) - (1.3)\right\} + (1.0)l + (1.2)l_2 + (1.3)l_3 \\ 0 &= +l_2\left\{p - (2) - \boxed{2} - (2.0) - (2.1) - (2.3)\right\} + (2.0)l + (2.1)l_1 + (2.3)l_3 \\ 0 &= +l_3\left\{p - (3) - \boxed{3} - (3.0) - (3.1) - (3.2)\right\} + (3.0)l + (3.1)l_1 + (3.2)l_2 \end{aligned} \qquad \textbf{(290)}$$

877. These equations determine the annual and sidereal motion of the nodes and inclinations of the orbits, and are precisely similar to those which determine the eccentricities and motions of the apsides, for if the terms depending on the displacement of the orbit of Jupiter be omitted, each satellite has four terms in latitude similar to the four equations of the centre, and arising like them from the changes in the positions of the orbits by the action of the matter at Jupiter's equator and their mutual attraction, they therefore depend on the inclinations and motions of the nodes of their own orbits, and on those of the other three. Hence, with the values of l, l_1, l_2, l_3, known by observation, these equations will give the four roots of p, the annual and sidereal motion of the nodes and the coefficients of the sixteen terms in the latitudes; for if it be assumed, that

$$l_1 = z_1 l; \quad l_2 = z_2 l; \quad l_3 = z_3 l,$$

these quantities will make l vanish from the preceding equations; the result will be four equations between z_1, z_2, z_3, and p, whence p will be obtained by an equation of the fourth degree.

Let p, p_1, p_2, p_3, be the roots of that equation, and let

$$z_1^{(1)},\ z_2^{(1)},\ z_3^{(1)};\quad z_1^{(2)},\ z_2^{(2)},\ z_3^{(2)};\quad z_1^{(3)},\ z_2^{(3)},\ z_3^{(3)};$$

be the values of $z_1,\ z_2,\ z_3,$ when p is successively changed to $p_1,\ p_2,\ p_3,$ they will give the coefficients required.

878. In article 861, it was assumed, that the latitudes of the satellites above the fixed plane were

$$s\ = l\ \sin(v\ + pt + \Lambda) \qquad s_/ = l_/ \sin(v_/ + pt + \Lambda)$$
$$s_2 = l_2 \sin(v_2 + pt + \Lambda) \qquad s_3 = l_3 \sin(v_3 + pt + \Lambda);$$

but if we refer them to the orbit of Jupiter, the term arising from the displacement of that orbit must be added to each, and if the different values of p be substituted, and the corresponding coefficients, the latitudes of the satellites above JN, the orbit of Jupiter at any time t, will be

$$\begin{aligned}
s\ &= +(l-1)q'\sin(v+y') \\
&\quad + l\ \sin(v + p\ t + \Lambda) \\
&\quad + l_1 \sin(v + p_1 t + \Lambda_1) \\
&\quad + l_2 \sin(v + p_2 t + \Lambda_2) \\
&\quad + l_3 \sin(v + p_3 t + \Lambda_3) \\
s_1 &= +(l_1-1)q'\sin(v_1+y') \\
&\quad + z_1\, l \sin(v_1 + pt + \Lambda) \\
&\quad + z_1^{(1)} l_1 \sin(v_1 + p_1 t + \Lambda_1) \\
&\quad + z_1^{(2)} l_2 \sin(v_1 + p_2 t + \Lambda_2) \\
&\quad + z_1^{(3)} l_3 \sin(v_1 + p_3 t + \Lambda_3) \\
s_2 &= +(l_2-1)q'\sin(v_2+y') \\
&\quad + z_2\, l \sin(v_2 + pt + \Lambda) \\
&\quad + z_2^{(1)} l_1 \sin(v_2 + p_1 t + \Lambda_1) \\
&\quad + z_2^{(2)} l_2 \sin(v_2 + p_2 t + \Lambda_2) \\
&\quad + z_2^{(3)} l_3 \sin(v_2 + p_3 t + \Lambda_3) \\
s_3 &= +(l_3-1)q'\sin(v_3+y') \\
&\quad + z_3\, l \sin(v_3 + pt + \Lambda) \\
&\quad + z_3^{(1)} l_1 \sin(v_3 + p_1 t + \Lambda_1) \\
&\quad + z_3^{(2)} l_2 \sin(v_3 + p_2 t + \Lambda_2) \\
&\quad + z_3^{(3)} l_3 \sin(v_3 + p_3 t + \Lambda_3).
\end{aligned} \tag{291}$$

879. The first term of each depends on the displacement of Jupiter's orbit, and the eight quantities

$$l,\ l_1,\ l_2,\ l_3, \qquad \Lambda,\ \Lambda_1,\ \Lambda_2,\ \Lambda_3,$$

are determined by observation; the first are the respective inclinations of the four satellites on Jupiter's orbit, and the last four are the longitudes of the nodes at the epoch. If it be required to find the latitude of the satellites above the fixed plane, it will be necessary to add to these the values of the latitudes, supposing the satellites to move on the orbit of Jupiter.

880. The inequalities in latitude which depend on the configuration of the bodies that acquire small divisors by integration are insensible, with the exception of those arising from the action of the sun depending on the angle $2v - 2U$. The part of the disturbing force whence these come is

$$R = \frac{3M^2}{4n^2}\left\{s^2\cos 2(v-U) - 4sS\cos(v-U) - \cos 2(v-U)\right\}$$

omitting the squares and products of S and s,[5]

$$\frac{dR}{dv} = \frac{3M^2}{2n^2}\sin 2(v-U)$$

$$\frac{dR}{ds} = \frac{3M}{2n^2}\left\{s\cos 2(v-U) - S\cos 2(v-U)\right\}$$

but

$$S = L'\sin(v + pt + \Lambda); \qquad S = l\sin(v + pt + \Lambda)$$

and

$$\frac{ds}{dv} = l\cos(v + pt + \Lambda),$$

and if $\frac{v}{n}$ be put for t, observing that

$$U = Mt = \frac{M}{n}v,$$

the equation in article 859 becomes

$$0 = \frac{d^2s}{dv^2} + N_/^2 s + \frac{3M^2}{2n^2}(L' - l)\sin\left(v - \frac{2M}{n}v - \frac{p}{n}v - \Lambda\right),$$

in which

$$N_/^2 = 1 + 2\frac{\left(r - \frac{1}{2}f\right)}{a^2} + \frac{3}{2}\frac{M^2}{n^2} + \sum m_/ a^2 a_/ B_1;$$

but, without sensible error,

$$N_{/}^{2} = 1 + \frac{\left(r - \frac{1}{2}f\right)}{a^{2}}.$$

In order to integrate this equation, let

$$s = K \cdot \sin\left(v - \frac{2M}{n}v - \frac{p}{n}v - \Lambda\right),$$

K being an indeterminate coefficient.

If that value of *s* be put in the equation, it will give

$$K = \frac{3M^{2}}{2n^{2}} \cdot \frac{L' - l}{\left(1 - 2\frac{M}{n} - \frac{p}{n}\right)^{2} - N_{/}^{2}}.$$

But in the divisor,

$$\left\{1 - \frac{2M}{n} - \frac{p}{n} + N_{/}\right\} \cdot \left\{1 - \frac{2M}{n} - \frac{p}{n} - N_{/}\right\};$$

$\frac{p}{n}$ is very small, and $N_{/}$ differs but little from unity; hence

$$1 - 2\frac{M}{n} - \frac{p}{n} + N_{/} = 2, \text{ nearly;}$$

therefore

$$s = -\frac{3M^{2}\left(L' - l\right)}{4n^{2}\left(2\frac{M}{n} + \frac{p}{n} + N_{/} - 1\right)} \sin\left(v - 2\frac{M}{n}v - \frac{p}{n}v - \Lambda\right);$$

a similar inequality exists for each root of *p*, including $\grave{p}$, which is the value of *p* depending on the displacement of the equator and orbit of Jupiter.

Now,

$$\frac{p}{n} + N_{/} = 1, \text{ nearly;}$$

consequently,

$$s = -\frac{3M}{8n}\left(L' - l\right).\sin\left(v - 2U - pt - \Lambda\right). \qquad \textbf{(292)}$$

Secular Inequalities of the Satellites, depending on the Variations in the Elements of Jupiter's Orbit

881. The secular inequalities in the elements of Jupiter's orbit, occasioned by the action of the planets, produce corresponding variations in the mean motions of the satellites, which, in the course of ages, will have a considerable effect on the theory of these bodies. These are obtained from

$$R=-\frac{S'r^2}{4D^3}\left\{1-3s^2-3S^2+12Ss\cos\left(U-v\right)\right\}$$
$$-\frac{\left(r-\tfrac{1}{2}f\right)}{r^3}\left(\tfrac{1}{3}-\left(s-s'\right)^2\right).$$

But, by articles 864 and 865,

$$s=g'\sin\left(v-t'\right),\quad S=g\sin\left(U-t\right),\quad s'=-q\sin\left(v+y\right);$$

and as the periodic inequalities are to be rejected,

$$\frac{S'}{D^3}=\frac{S'}{D'^3}\left\{1+3\left(\frac{DdD}{D'^2}\right)^2\right\}=M^2\left\{1+3H^2\sin^2\left(Mt+E-\Pi\right)\right\}=M^2\left(1+\tfrac{3}{2}H^2\right).$$

For the same reason,

$$r^2=a^2\left(1+\tfrac{1}{2}e^2-e\cos\left(nt+\epsilon-v\right)\right)^2=a^2\left(1+\tfrac{3}{2}e^2\right);$$

and if it be observed that

$$\frac{3a^3nM^2}{4}=\boxed{0};\qquad\frac{\left(r-\tfrac{1}{2}f\right)}{a^2}n=(0),$$

the value of *anR* is

$$anR=-\tfrac{1}{2}\boxed{0}\left\{e^2+H^2-g^2+2gg_{/}\cos\left(t'-t\right)-g_{/}^2\right\}-\tfrac{1}{2}(0)\left\{q^2+2qg_{/}\cos\left(t'+y\right)+g_{/}^2-e^2\right\}.$$

If this quantity be put in equation (259), the result will be

$$\frac{d.dv'}{dt}=-2\boxed{0}\left\{e^2+H^2-g^2+2gg'\cos\left(t'-t\right)-g'^2\right\}+3\left(0\right)\left\{q^2+2qg'\cos\left(t+y\right)+g'^2-e^2\right\}.$$

This, however, only gives the inequalities on the orbit; but its projection on the fixed plane, by article 548, is[6]

$$d=dv\left\{1+\tfrac{1}{2}s^2-\tfrac{1}{2}\frac{ds^2}{dv^2}\right\}.$$

Now,

$$s=g'\sin\left(v-t'\right)=g'\sin v\cos t'-g'\cos v\sin t'.$$

The substitution of this quantity, and of its differential, gives

$$dv' = dv + \tfrac{1}{2}\left\{g'\cdot\sin t'\frac{d(g'\cos t')}{dt} - g'\cdot\cos t'\frac{d(g'\sin t')}{dt}\right\},$$

the value of dv' projected on the fixed plane; therefore

$$\begin{aligned}\frac{d\cdot dv}{dt} &= +\tfrac{1}{2}\left\{g'\sin t'\frac{d(g'\cos t')}{dt} - g'\cos t'\frac{d(g'\sin t')}{dt}\right\}\\ &\quad -2\boxed{0}\left\{e^2 + H^2 - g^2 + 2gg'\cos(t'-t) - g'^2\right\}\\ &\quad +\tfrac{1}{2}(0)\left\{e^2 - q^2 - 2qg'\cos(t'+y) - g'^2\right\}.\end{aligned} \qquad \textbf{(293)}$$

Since all the quantities $g\sin t$, $g\cos t$, $g_{/}\sin t_{/}$, $g_{/}\cos t'$, y and q, are given in the preceding articles, it may be found that

$$\begin{aligned}\frac{d\cdot dv}{dt} &= +4.\boxed{0}.(1-l)^2.\grave{}Lbt - 6.(0).\ l^2.\grave{}L.bt\\ &\quad +(1-l).\ l.\left\{(0)+\boxed{0}+(0.1)+(0.2)+(0.3)\right\}.\grave{}L.bt\\ &\quad -\tfrac{1}{2}l(0)\cdot\grave{}L\cdot bt - \tfrac{1}{2}(1-l)\cdot\boxed{0}\cdot\grave{}L\cdot bt\\ &\quad +\tfrac{1}{2}(0.1)\cdot\left\{(l-1)l' + (l_1-1)\,l\right\}\cdot\grave{}L\cdot bt\\ &\quad +\tfrac{1}{2}(0.2)\left\{(l-1)\,l_2 + (l_2-1)\,l\right\}\cdot\grave{}L\cdot bt\\ &\quad +\tfrac{1}{2}(0.3)\left\{(l-1)\,l_3 + (l_3-1)\,l\right\}\grave{}L\cdot bt - 2\boxed{0}H^2.\end{aligned}$$

But in consequence of the relations between l, l_1, l_2, l_3, in article 859,

$$\frac{d\cdot dv}{dt} = 4(1-l)^2.\boxed{0}.\grave{}Lbt - 6(0)\,l^2\grave{}Lbt - 2H^2.\boxed{0}.$$

In considering the action of Saturn only, equations (204) give the numerical value of H; to abridge, if $\bar{e}$ be the value of H at the epoch, then

$$H = \bar{e} + ct + \&c.;$$

and omitting the square of the time,

$$H^2 = 2\bar{e}ct,$$

and the integral becomes

$$dv = 2(1-l)^2\boxed{0}\cdot\grave{}Lbt^2 - 3(0)\,l^2\grave{}Lbt^2 - 2\bar{e}ct^2\boxed{0}. \qquad \textbf{(294)}$$

This inequality in the mean motion of m varies with the eccentricity of the orbit of Jupiter, and is similar to the acceleration in the mean motion of the moon, but it will not be perceptible for many years, nor has it hitherto been perceived.

882. If there was but one satellite, the first of equations (285) would give

$$l = \frac{\boxed{0}}{\boxed{0}+(0)}.$$

In the theory of the moon, $\boxed{0}$ is vastly greater than (0), so that $l = 1 - \frac{(0)}{\boxed{0}}$ differs but little from unity, which reduces the equation (294) to $dv = -2\boxed{0}\bar{e}\,.ct^2$, where $\bar{e}$ is the eccentricity of the earth's orbit at the epoch; and substituting $\frac{3}{4}\frac{M}{n}$ for $\boxed{0}$, it becomes

$$dv = -\frac{3}{2}\frac{M^2}{n}\bar{e}ct^2;$$

which is the same with the acceleration of the moon.

883. One secular variation alone is sensible at present, and that only in the mean motion of the fourth satellite; it is derived from equation (293), each term of which must be determined separately. When e^2 and H^2 are omitted, its second term[7]

$$\tfrac{1}{2}\left\{g'^2 - 2gg'\cos(t'-t) + g^2\right\}$$

is the square of that part of the latitude of the satellite m above the orbit of Jupiter, which is independent of v; therefore the expression is equal to the square of s in article 861, where v is omitted; but as l, $l_{/}$, &c., are very small, their squares and products may be neglected, so that the quantity required, after the reduction of the products of the sines to the cosines of the differences of the arcs, is

$$g^2 - 2gg'\cos(t'-t) + g'^2 = (1-l)^2 q'^2 + 2(l-1)q'\left\{l\cos(pl + \Lambda - y') + l_{/}\cos(p_{/}l + \Lambda' - y') + \&\text{c.}\right\}$$

hence

$$\frac{d\,.dv}{dt} = 4(l-1)q'\boxed{0}\left\{l\cos(pt + \Lambda - y') + \&\text{c.}\right\}.$$

Again,

$$q^2 + 2qg'\cos(t' + y) + g'^2,$$

the third term of equation (293), is the square of the latitude of m above the equator of Jupiter, when v is omitted, and is therefore equal to the square of

$$l q' \sin y' + l \sin(pt + \Lambda) + l_{/} \sin(p_{/} t + \Lambda_{/}) + \&c.$$

which is given by the first of equation (291). Whence

$$\frac{d\,.dv}{dt} = -6(0)\, l q' \{ l \cos(pt + \Lambda - y') + \&c.\}.$$

In the third place the same expression of s gives

$$g' \sin t' = +(l - 1) q' \cos y' + l \cos(pt + \Lambda) + \&c. - l \cos t$$
$$g' \cos t' = -(l - 1) q' \sin y' - l \sin(pt + \Lambda) - \&c. - l \sin t.$$

By means of these values the first term of equation (293) becomes

$$\frac{ddv}{dt} = -\tfrac{1}{2}(l - 1).q' \{ pl \cos(pt + \Lambda - y') + \&c.\}$$

When these three parts are added, they constitute the whole of equation (293), the integral of which is

$$d v = -\{6(0)\, l + 4(1 - l)\boxed{0}\} q' \left\{ \frac{l}{p} \sin(pt + \Lambda - y') + \frac{l_{/}}{p_{/}} \sin(p_{/} t + \Lambda_{/} - y') + \&c. \right\}$$
$$+ \tfrac{1}{2}(1 - l) q' \{ l \sin(pt + \Lambda - y') + l_{/} \sin(p_{/} t + \Lambda_{/} - y') + \&c.\}$$

The only part that has a sensible effect is

$$d v = -\frac{\{4(1 - l_3)\boxed{3} - \tfrac{1}{2}(1 - l_3) p_3 + 6(3)\, l_3\}}{p_3} q' l_3 \times \sin(p_3 + \Lambda_3 - y'), \qquad \textbf{(295)}$$

and that in the motions of the fourth satellite only.

884. With regard to the moon, $l = 1 - \frac{(0)}{\boxed{0}}$ differs but little from unity, and $p = \boxed{0}$ nearly; hence, for that body,

$$d v = -\frac{19}{2} \cdot (0) \cdot \frac{q' l}{p} \sin(v + pt - y'),$$

which coincides with equation (244), supposing the obliquity of the ecliptic to be very small.

Notes

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[1] The remaining three chapters (II, III, and IV) in Book IV are numbered VII, VIII, and IX in the 1st edition.
[2] This *reads* "The equations in article 277" in the 1st edition.
[3] This is the first instance of the use of the back-prime syntax in the text.
[4] The arguments of both sine functions *read* $\left(p\grave{t}+\grave{}\Lambda\right)$ in the 1st edition.
[5] The 2nd term in the 2nd equation *reads* $-2S\cos(\cos(v-U)$ in the 1st edition.
[6] The parentheses are mismatched. The expression *reads* $d = dv(1+\frac{1}{2}s^2-\frac{1}{2}\frac{ds^2}{dv^2}\}$ in the 1st edition.
[7] The middle term *reads* $-2gg^{2}\cos\left(t'-t\right)$ in the 1st edition.