BOOK IV

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CHAPTER I

THEORY OF JUPITER'S SATELLITES

798. JUPITER is attended by four satellites, which were discovered by Galileo[1] on the 1st of June, 1610; their orbits are nearly in the plane of Jupiter's equator, and they exhibit all the phenomena of the solar system, on a small scale and in short periods. The eclipses of these satellites afford the easiest method of ascertaining terrestrial longitudes; and the frequency of the occurrence of an eclipse renders the theory of their motions nearly as important to the geographer as that of the moon.

799. The orbits of the two first satellites are circular, subject only to such eccentricities as arise from the disturbing forces; the third and fourth satellites have elliptical orbits; the eccentricity however is so small, that their elliptical motion is determined along with those perturbations that depend on the eccentricities of the orbits.

800. Although Jupiter's satellites might be regarded as an epitome of the solar system, they nevertheless require a new investigation, on account of the nearly commensurable ratios in the mean motions of the three first satellites, the action of the sun, the ellipticity of Jupiter's spheroid, and the displacement of his orbit by the action of the planets.

801. It appears, from observation, that the mean motion of the first satellite is nearly equal to twice that of the second; and that the mean motion of the second is nearly equal to twice that of the third; whence the mean motion of the first, minus three times that of the second, plus twice that of the third, is zero; but the last ratio is so exact, that from the earliest observations it has always been zero. It is also found that, from the time of the discovery of the satellites, the mean longitude of the first, minus three times that of the second, plus twice that of the third, is equal to 180°: and it will be shown, in the theory of these bodies, that even if these ratios had not been exact in the origin of their motions, their mutual attractions would have made them so. They are the cause of the principal inequalities in the longitude of the satellites; and as they exist also in their synodic motions, they have a great influence on the times of their eclipses, and indeed on their whole theory.

802. The prominent matter at Jupiter's equator, together with the action of the satellites themselves, causes a direct motion in the apsides, which changes the relative position of the orbits, and alters the attractive force of the satellites; consequently each satellite has virtually four equations of the centre, or rather, that part of the longitude of each satellite that depends on the eccentricity, consists of four principal terms; one that arises from the true ellipticity of its own orbit, and three others, depending on the positions of the apsides of the other three orbits. Inequalities perfectly similar to these are produced in the radii vectores by the same cause, consisting of the same number of terms, and depending on the same quantities.

803. Astronomers imagined that the orbits of the satellites had a constant inclination to the plane of Jupiter's equator; however, they have not always the same inclination, either to the plane of his equator or orbit, but to certain imaginary fixed planes passing between these, and also through their intersection.

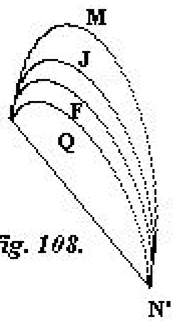


fig. 108.

Let NJN′ be the orbit of Jupiter, NQN′ the plane of his equator extended so as to cut his orbit in NN′; then, if NMN′ be the orbit of a satellite, it will always preserve very nearly the same inclination to a fixed plane NFN′, passing between the planes NQN′ and NJN′, and through the line of their nodes. But although the orbit of the satellite preserves nearly the same inclination to NFN′, its nodes have a retrograde motion on that plane. The plane FN itself is not absolutely fixed, but moves slowly with the equator and orbit of Jupiter. Each satellite has a different fixed plane, which is less inclined to the plane of Jupiter's equator the nearer the satellite is to the planet, evidently arising from the attraction of the protuberance at Jupiter's equator, which retains the satellites nearly in the plane of the equator; furnishing another proof of the mutual attraction of the particles of matter.

804. The equatorial matter of Jupiter's spheroid causes a retrograde motion in the nodes of the orbits of the satellites; which alters their mutual attraction, by changing the relative position of their planes, so that the latitude of any one satellite not only depends on the position of the node of its own orbit, but on the nodes of the other three; and as the position of Jupiter's equator is perpetually varying, in consequence of the action of the sun and satellites, the latitude of these bodies varies also with the inclination of Jupiter's equator on his orbit, and the position of its nodes. Thus, the principal inequalities of the satellites arise from the compression of Jupiter's spheroid, and from the direct and indirect action of the sun and satellites themselves.

805. The secular variation in the form and position of Jupiter's orbit is the cause also of secular variations in the motions of the satellites, similar to those in the motions of the moon occasioned by the variation in the eccentricity and position of the earth's orbit.

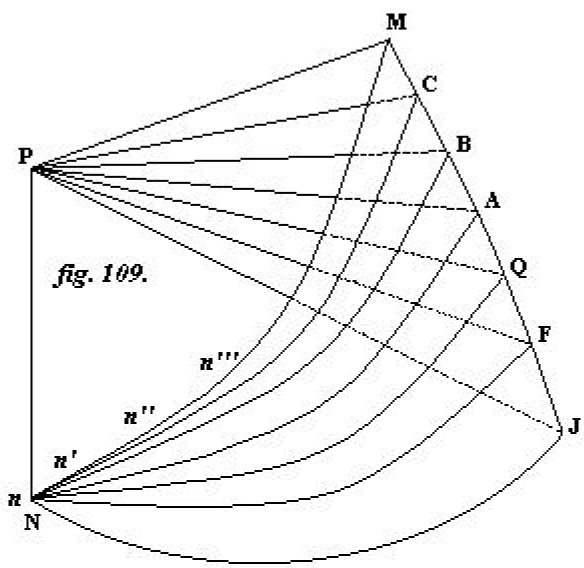


fig. 109.

806. The position of the orbit of a satellite may be known by supposing five planes, of which FN, passing between JN and QN, the planes of Jupiter's orbit and equator, always retains very nearly the same inclination to them. The second plane An moves uniformly on FN, retaining nearly the same inclination on it. The third Bn' moves in the same manner on An; the fourth Cn'' moves similarly on Bn'; and the fifth Mn''', which has the same kind of motion on Cn'', is the orbit of the satellite. The motion of the nodes are retrograde, and each satellite has set of planes peculiar to itself. In conformity with this, the latitude of a satellite above the variable orbit of Jupiter, is expressed by five terms; the first of which is relative to the displacement of the orbit and equator of Jupiter; the second is relative to the inclination of the orbit of the satellite on its fixed plane; and the other three terms depend on the position and motion of the nodes of the other three orbits. The inequalities which have small divisions, arising from the configuration of the bodies, are insensible in latitude, with the exception of those produced by the sun, which modify the preceding quantities.

807. For the solution of the problem of the satellites, the data that must be determined by observation for a given epoch, are, the compression of Jupiter's spheroid, the inclination of his equator on his orbit, the longitude of its nodes, the eccentricity of his orbit, its position, and its secular variations; the masses of the four satellites, their mean distances, periodic times, the eccentricities and inclinations of their orbits, together with the longitude of their apsides and nodes. The masses of the satellites and the compression of Jupiter are determined from the inequalities of the satellites themselves.

808. The orbits of the four satellites may be regarded as circular, because the eccentricity of the third, and even the fourth, is so small, that their equations of the centre will be determined with the perturbations depending on the eccentricities and inclinations. Thus, with regard to the two first, and nearly for the other two, the true longitude is the sum of the mean longitude and perturbations; and the radius vector will be found by adding the perturbations to the mean distance.

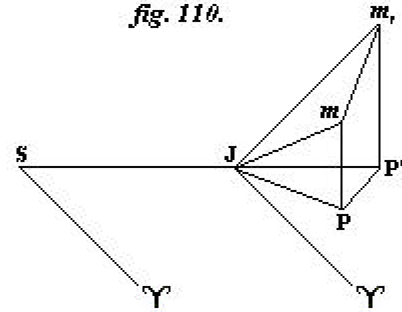
809. A satellite *m* is troubled by the other three, by the sun, and by the excess of matter at Jupiter's equator. The problem however will be limited to the action of the sun, of Jupiter's spheroid, and of one satellite; the resulting equations will be general, from whence the action of each body may be computed separately, and the sum will be the effect of the whole.

810. Let m and $m_/$ be the masses of any two satellites, x, y, z, x', y', z', their rectangular co-ordinates referred to the centre of gravity of Jupiter, supposed to be at rest; r, r' their radii vectores; then the disturbing action of $m_/$ on m is

$$\frac{m_/\left(xx'+yy'+zz'\right)}{r_/^3}-\frac{m_/}{\sqrt{\left(x'-x\right)^2+\left(y'-y\right)+\left(z'-z\right)^2}}=R;$$

Consequently the sign of R must be changed in equations (155) and (156), since it is assumed to be negative in this case.

fig. 110.



The satellites move nearly in the plane of Jupiter's equator, which in 1750 was inclined to the plane of his orbit at an angle of $3^\circ\,5'\,30''$; and as the fixed planes pass between these two, the inclinations of the orbits of the satellites to them are very small; consequently $s = mP$, $s_/ = m_/P$, fig. 110, the tangents of the latitude of the two satellites on PJP', the fixed plane of m, are very small.

If g be the vernal equinox of Jupiter, the longitudes of the two satellites are $g\,JP = v$, $g\,JP' = v_/$, and therefore

$$x=\frac{r\cos v}{\sqrt{1+s^2}},\quad y=\frac{r\sin v}{\sqrt{1+s^2}},\quad x=\frac{rs}{\sqrt{1+s^2}}.$$

If x', y', z', the co-ordinates of $m_/$, be equal to the same quantities accented, the action of $m_/$ on m, expressed in polar co-ordinates, will be

$$R=+\frac{m_/ r}{r_/^2}\left\{ss_/+\left(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s_/^2\right)\cos\left(v_/-v\right)\right\}-\frac{m_/}{\sqrt{r^2-2rr_/\cos\left(v_/-v\right)+r_/^2}}$$
$$-\frac{m_/\cdot rr_/\cdot\left\{ss_/-\tfrac{1}{2}\left(s^2+s_/^2\right)\cos\left(v_/-v\right)\right\}}{\left\{r^2-2rr_/\cdot\cos\left(v_/-v\right)+r_/^2\right\}^{\frac{3}{2}}},$$

when $s^4,\ s_/^4$ are neglected.

811. If S' be the mass of the sun, and X', Y', Z', his co-ordinates, his action upon m will be expressed by

$$R=\frac{S'\left(X'x+Y'y+Z'z\right)}{D^3}-\frac{S'}{\sqrt{\left(X'-x\right)^2+\left(Y'-y\right)^2+\left(Z'-z\right)^2}},$$

D being his distance from the centre of Jupiter.

Let Jupiter and his orbit be assumed to be at rest, and let his motion be referred to the sun, which is the same as supposing the sun to move in the orbit of Jupiter with the velocity of that planet; if S be the tangent of the sun's latitude above the fixed plane PJP', and $U=g\,SJ$, his longitude seen from the centre of Jupiter when at rest, then

$$X'=\frac{D\cos U}{\sqrt{1+S^2}},\quad Y'=\frac{D\sin U}{\sqrt{1+S^2}},\quad Z'=\frac{D\,.\,S}{\sqrt{1+S^2}},$$

and[2]

$$R=-\frac{S'}{D}-\frac{S'r^2}{4D^3}\left\{1-3s^2-3S^2+12sS\left(\cos\left(U-v\right)+3\cos 2\left(U-v\right)\right)\right\},$$

which is the action of the sun on the satellite when terms divided by D^4 are omitted, for the distance of the satellite from Jupiter is incomparably less than the distance of Jupiter from the sun.

812. The attraction of the excess of matter at Jupiter's equator is expressed by

$$R=-\left(r-\tfrac{1}{2}f\right)\left(\tfrac{1}{3}-n^2\right)\cdot\frac{J\,.\,R`^2}{r^3},$$

in which n is the sine of the declination of the satellite on the plane of Jupiter's equator; J the mass of Jupiter; $2R`$ his equatorial diameter; $r$ his ellipticity, and $f$ the ratio of the centrifugal force to gravity at his equator. Now it may be assumed that $J = 1$, $R` = 1$; and if s' be the tangent of the latitude that the satellite would have above the fixed plane if it moved in the plane of Jupiter's equator, and as s is its latitude above that plane, when moving in its own orbit, $n = s - s'$ nearly; hence

$$R=-\frac{\left(r-\tfrac{1}{2}f\right)}{r^3}\left\{\tfrac{1}{3}-\left(s-s'\right)^2\right\}.$$

813. Thus the whole force that troubles the motion of m is

$$\begin{aligned}R=&+\frac{m_{/}r}{r_{/}^{2}}\left\{ss_{/}+\left(1-\tfrac{1}{2}s^2-\tfrac{1}{2}s_{/}^{2}\right)\cos\left(v_{/}-v\right)\right\}-\frac{m_{/}}{\sqrt{r^2-2rr_{/}\cos\left(v_{/}-v\right)+r_{/}^{2}}}\\&-\frac{m_{/}rr_{/}\left\{ss_{/}-\tfrac{1}{2}\left(s^2+s_{/}^{2}\right)\cos\left(v_{/}-v\right)\right\}}{\left\{r^2-2rr_{/}\cos\left(v_{/}-v\right)+r_{/}^{2}\right\}^{\frac{3}{2}}}-\frac{S'}{D}-\frac{S'r^2}{4D^3}\left\{1-3s^2-3S^2+12sS\cos\left(U-v\right)+3\cos 2\left(U-v\right)\right\}\\&-\frac{\left(r-\tfrac{1}{2}f\right)}{r^3}\left\{\tfrac{1}{3}-\left(s-s'\right)^2\right\}.\end{aligned}$$

814. If the squares of S, s, and s' be omitted, the only force that troubles the satellites in longitude and distance is

$$R=+\frac{m_{/}r}{r_{/}^{2}}\cos\left(v_{/}-v\right)-\frac{m_{/}}{\sqrt{r^2-2rr_{/}\cos\left(v_{/}-v\right)+r_{/}^{2}}}-\frac{S'}{D}-\frac{S'r^2}{4D^3}\left\{1+3\cos 2\left(U-v\right)\right\}-\frac{\left(r-\tfrac{1}{2}f\right)}{3r^3}.$$

When the eccentricities are omitted, the radii vectores, r and r', become a, $a_{/}$, half the greater arcs of the orbits, and that part of R that depends on the mutual attraction of the satellites, is

$$R'=\frac{m_{/}a}{a_{/}^{2}}\cos\left(n_{/}t-nt+\in_{/}-\in\right)-\frac{m_{/}}{\sqrt{a^2-2aa_{/}\cos\left(n_{/}t-nt+\in_{/}-\in\right)+a_{/}^{2}}}$$

$nt+\in$, $n_{/}t+\in_{/}$, being the mean longitudes of m and $m_{/}$. This expression may be developed into the series

$$R' = m_{/}\left\{\tfrac{1}{2}A_0 + A_1\cdot\cos\left(n_{/}t - nt + \in_{/} - \in\right) + A_2\cos 2\left(n_{/}t - nt + \in' - \in\right) + \&\text{c.}\right\}$$

This is the part of R that is independent of the eccentricities, and is identical with the series in article 446; therefore the coefficients $A_0,\ A_1,\ \&\text{c.},$ and their differences, may be computed by the same formulae as for the planets, observing to substitute $A_{/} - \dfrac{a}{a_{/}^{2}}$ for $A_{/}$.

But, by article 445,

$$r = a(1+u) \qquad r_{/} = a_{/}(1+u_{/})$$
$$v = nt + \in + v' \qquad v_{/} = n_{/}t + \in_{/} + v'_{/},$$

where $u,\ u_{/},\ v',\ v'_{/},$ are the elliptical parts of the radii vectores, and of the longitudes of m and $m_{/}$. By the same article, the general formula for the development of R, according to the powers and products of these minute quantities, is

$$R = R' + au_{/}\cdot\frac{dR'}{da} + a_{/}u_{/}\cdot\frac{dR'}{da_{/}} + \left(v'_{/} - v'\right)\frac{dR'}{ndt} + \&\text{c.}$$

From the preceding value of R' the quantities $\dfrac{dR'}{da},\ \dfrac{dR'}{da_{/}},\ \&\text{c.},$ may be found; and, when substituted, it will be seen afterwards that the only requisite part of R is

$$\begin{aligned}
R = &+ m_{/}\left\{\tfrac{1}{2}A_0 + A_1\cdot\cos\left(n_{/}t - nt + \in_{/} - \in\right) + A_2\cos 2\left(n_{/}t - nt + \in' - \in\right) + \&\text{c.}\right\}\\
&+ \frac{m_{/}}{2}\cdot au\cdot\frac{dA_0}{da} + m_{/}au\frac{dA_2}{da}\cdot\cos 2\left(n_{/}t - nt + \in_{/} - \in\right)\\
&+ m_{/}a_{/}u'\frac{dA_1}{da_{/}}\cdot\cos\left(n_{/}t - nt + \in_{/} - \in\right)\\
&- m_{/}v'_{/}A_1\cdot\sin\left(n_{/}t - nt + \in_{/} - \in\right)\\
&+ 2m_{/}v'A_2\cdot\sin 2\left(n_{/}t - nt + \in_{/} - \in\right).
\end{aligned}$$

Because the satellites move in nearly circular orbits, $u,\ u_{/},\ v',$ and $v'_{/},$ may be regarded as variations arising either entirely from the disturbing forces, as in the first and second satellites, or from that force conjointly with a real but very small ellipticity, as in the third and fourth; therefore

$$r = a(1+du), \qquad r_{/} = a_{/}(1+du_{/})$$
$$v = nt + \in + dv, \qquad v_{/} = n_{/}t + \in_{/} + dv_{/}$$

Now, $r = a(1+u)$ gives $r^2 = a^2(1+2u)$; for u is so small, that its square may be omitted; hence $du = \frac{rdr}{a^2}$: consequently $du_{/} = \frac{r_{/}dr_{/}}{a_{/}^2}$; and when R = 0, equation (156) gives, for the elliptical part of rdr only,

$$dv = \frac{2d(rdr)}{a^2 . ndt}, \text{ and } dv_{/} = \frac{2d(r_{/}dr_{/})}{a_{/}^2 . n_{/}dt},$$

when the squares of the eccentricities are omitted.

815. If these quantities be substituted in R instead of u, $u_{/}$, v', and $v'_{/}$, it becomes

$$\begin{aligned} R = &+m_{/}\left\{\tfrac{1}{2}A_0 + A_1.\cos(n_{/}t - nt + \epsilon_{/} - \epsilon) + A_2\cos 2(n_{/}t - nt + \epsilon' - \epsilon) + \&c.\right\} \\ &+\frac{m_{/}}{2}\cdot\frac{rdr}{a^2}\cdot a\left(\frac{dA_0}{da}\right) \\ &+m_{/}\cdot\frac{rdr}{a^2}\cdot a\left(\frac{dA_2}{da}\right)\cdot\cos 2(n_{/}t - nt + \epsilon_{/} - \epsilon) \\ &+m_{/}\cdot\frac{r_{/}dr_{/}}{a_{/}^2}\cdot a_{/}\left(\frac{dA_1}{da_{/}}\right)\cdot\cos(n_{/}t - nt + \epsilon_{/} - \epsilon) \\ &+4m_{/}\cdot\frac{d(rdr)}{a^2 \cdot ndt}\cdot A_2 .\sin 2(n_{/}t - nt + \epsilon_{/} - \epsilon) \\ &-2m_{/}\cdot\frac{d(r_{/}dr_{/})}{a_{/}^2 . n_{/}dt}\cdot A_1 .\sin(n_{/}t - nt + \epsilon_{/} - \epsilon) \\ &+\&c. \end{aligned} \qquad \textbf{(253)}$$

816. If $\frac{S'}{D}$ and the eccentricity be omitted, the action of the sun on m is

$$R = -\frac{S'a^2}{4D'^3}\{1 + 3\cos 2(Mt - nt + E - \epsilon)\};$$

where D' is the mean distance of Jupiter from the sun, and $Mt+E$ his mean longitude referred to the sun. In the troubled orbit a, $nt+\epsilon$, and D' become

$$a\left(1 + \frac{rdr}{a^2}\right), \quad nt + \epsilon - \frac{2d(rdr)}{a^2 \cdot ndt}, \quad \text{and } D'\left(1 - \frac{DdD}{D'^2}\right);$$

and as, by article 383, $\frac{S'}{D'^3} = M^2$, when the mass of Jupiter is omitted in comparison of that of the sun, the whole disturbing action of the sun is

$$\begin{aligned} R = &-\frac{M^2a^2}{4} - \frac{M^2}{2}\cdot r\delta r - \frac{3M^2a^2}{4}\cdot\cos 2\left(nt - Mt + \in - E\right) \\ &-\tfrac{3}{4}M^2a^2\cdot\frac{D\delta D}{D'^2} - M^2\cdot\frac{6r\delta r}{4}\cdot\cos 2\left(nt - Mt + \in - E\right) \\ &+3M^2\cdot\frac{d\left(r\delta r\right)}{ndt}\cdot\sin 2\left(nt - Mt + \in - E\right) \end{aligned} \qquad \textbf{(254)}$$

when the squares of the eccentricities are omitted.

817. In the same manner it is easy to see that the effect of Jupiter's compression is

$$R = -\frac{\left(r - \frac{1}{2}f\right)}{3a^3} + \frac{\left(r - \frac{1}{2}f\right)}{a^5}\cdot r\delta r\,.$$

The three last values of R contain all the forces that trouble the longitude and radius vector of m.

FIRST APPROXIMATION

Perturbations in the Radius Vector and Longitude of m that are independent of the Eccentricities

818. Since R has been taken with a negative sign, equation (155) becomes

$$\frac{d^2.r\delta r}{dt^2} + m\cdot\frac{r\delta r}{r^3} + 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right) = 0\,. \qquad \textbf{(255)}$$

The mass of each satellite is about ten thousand times less than the mass of Jupiter, and may therefore be omitted in the comparison, and if Jupiter be taken as the unit of mass $m = 1$.

When the eccentricity is omitted $r = a$; but by article 556 the action of the disturbing forces produces a permanent increase in a, which may be expressed by δa, therefore if $\left(a + \delta a\right)^{-3}$ be put for r^{-3},

$$\frac{d^2.r\delta r}{dt^2} + \frac{r\delta r}{a^3}\left(1 - 3\frac{\delta a}{a}\right) + 2\int \mathrm{d}R + r\left(\frac{dR}{dv}\right) = 0\,. \qquad \textbf{(256)}$$

819. When the eccentricities are omitted,

$$R = +m_{/}\left\{\tfrac{1}{2}A_0 + A_1\cos\left(n_{/}t - nt + \in_{/} - \in\right) + A_2\cos 2\left(n_{/}t - nt + \in_{/} - \in\right)\right\}$$

$$
\begin{aligned}
&+\frac{m_{/}}{2}\cdot\frac{rd\,r}{a^2}\cdot a\left(\frac{dA_0}{da}\right)\\
&-\tfrac{1}{4}M^2a^2-\tfrac{1}{2}M^2\cdot\frac{rd\,r}{a^2}-\tfrac{3}{4}M^2a^2\cos 2(nt-Mt+\in-E)\\
&-\frac{\left(r-\frac{1}{2}f\right)}{3a^3}+\frac{\left(r-\frac{1}{2}f\right)}{a^5}\cdot rd\,r.
\end{aligned}
\qquad \textbf{(257)}
$$

Since d*R* relates to the mean motion of *m*, the term

$$\frac{m_{/}}{2}\cdot\frac{rd\,r}{a^2}\cdot a\left(\frac{dA_0}{da}\right)$$

gives

$$2\int \mathrm{d}R=m_{/}\cdot\frac{rd\,r}{a}\cdot\left(\frac{dA_0}{da}\right);$$

moreover the same term gives

$$r\left(\frac{dR}{dr}\right)=\frac{m_{/}}{2}\cdot\frac{rd\,r}{a}\left\{\frac{dA_0}{da}+a\frac{d^2A_0}{da^2}\right\}.$$

With regard to Jupiter's compression

$$\int \mathrm{d}R=R,\quad r\left(\frac{dR}{dr}\right)=-3R,$$

consequently

$$2\int \mathrm{d}R+r\left(\frac{dR}{dr}\right)=\frac{\left(r-\frac{1}{2}f\right)}{3a^3}-\frac{\left(r-\frac{1}{2}f\right)}{a^5}\cdot rd\,r.$$

Attending to these circumstances, and observing that

$$\frac{1}{a^3}=n^2=\frac{1+m}{a^3},$$

it will be found, when the eccentricities are omitted and the whole divided by a^2, that

$$
\begin{aligned}
&+\frac{d^2\,.\,rd\,r}{a^2dt^2}+N^2\cdot\frac{rd\,r}{a^2}+2n^2K+n^2\cdot\frac{r-\frac{1}{2}f}{3a^2}-M^2\\
&+\sum.\frac{m_{/}n^2}{2}\cdot a^2\left(\frac{dA_0}{da}\right)-3M^2\cdot\frac{2n-M}{2n-2M}\cdot\cos 2(nt-Mt+\in-E)\\
&+\sum.m_{/}n^2\cdot\left\{a^2\left(\frac{dA_1}{da}\right)+\frac{2n}{n-n_{/}}\cdot aA_1\right\}.\cos(n_{/}t-nt+\in_{/}-\in)
\end{aligned}
\qquad \textbf{(258)}
$$

$$+\sum .m_{/}n^2 \cdot \left\{ a^2 \left(\frac{dA_2}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_2 \right\} \cdot \cos 2\left(n_{/}t - nt + \in_{/} - \in \right)$$
$$+ \&c.\ \&c. = 0.$$

Where to abridge

$$N^2 = n^2 \left\{ 1 - \frac{3da}{a} - \frac{\left(r - \frac{1}{2}f \right)}{a^2} - \frac{2M^2}{n^2} + \sum \frac{m_{/}a^2}{2} \left\{ 3\left(\frac{dA_0}{da} \right) + a\left(\frac{d^2A_0}{da^2} \right) \right\} \right\};$$

a quantity that differs little from n^2, for the last term is extremely small in consequence of the factor $m_{/}$: the variation of the mean distance a is very small, and so are the other two parts depending on the compression of Jupiter and the action of the sun. Indeed M and $N-n$ may be omitted, in comparison of n in the terms arising from the action of the sun after integration, for the motion of Jupiter is much slower than the motion of his satellites.

820. The preceding equation may be integrated by the method of indeterminate coefficients, if it be assumed that

$$\frac{rdr}{a^2} = B + m_{/}b\cos\left(n_{/}t - nt + \in_{/} - \in \right) + m_{/}b_{(1)}\cos 2\left(n_{/}t - nt + \in_{/} - \in \right) + \&c. + Gm_{/}\cos 2\left(nt - Mt + \in - E \right).$$

For a comparison of the coefficients of similar cosines after the substitution of this quantity and its differential gives

$$B = -\frac{n^2}{N^2} \left\{ 2K + \frac{\left(r - \frac{1}{2}f \right)}{3a^2} - \frac{M^2}{n^2} + \sum \frac{m_{/}}{2} a^2 \left(\frac{dA_0}{da} \right) \right\}$$

$$b = \frac{\left\{ a^2 \left(\frac{dA_1}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_1 \right\} n^2}{\left(n-n_{/} \right)^2 - N^2}$$

$$b_{(1)} = \frac{\left\{ a^2 \left(\frac{dA_2}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_2 \right\} n^2}{4\left(n-n_{/} \right)^2 - N^2}$$

$$b_{(2)} = \frac{\left\{ a^2 \left(\frac{dA_3}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_3 \right\} n^2}{9\left(n-n_{/} \right)^2 - N^2} \&c.$$

$$G = -\frac{M^2}{n^2},$$

and the integral of (258) is

$$\frac{rdr}{a^2}=-\frac{n^2}{N^2}\left\{2K+\frac{\left(r-\frac{1}{2}f\right)}{3a^2}-\frac{M^2}{n^2}+\Sigma\frac{m_{/}}{2}a^2\left(\frac{dA_0}{da}\right)\right\}$$

$$-\frac{M^2}{n^2}\cos2(nt-Mt+\in-E)$$

$$+\Sigma m'\left\{\begin{array}{l}\dfrac{n^2}{(n-n_{/})^2-N^2}\left\{a^2\left(\dfrac{dA_1}{da}\right)+\dfrac{2n}{n-n_{/}}aA_1\right\}\cos(n_{/}t-nt+\in_{/}-\in)\\ \dfrac{n^2}{4(n-n_{/})^2-N^2}\left\{a^2\left(\dfrac{dA_2}{da}\right)+\dfrac{2n}{n-n_{/}}aA_2\right\}\cos2(n_{/}t-nt+\in_{/}-\in)\\ \dfrac{n^2}{9(n-n_{/})^2-N^2}\left\{a^2\left(\dfrac{dA_3}{da}\right)+\dfrac{2n}{n-n_{/}}aA_3\right\}\cos3(n_{/}t-nt+\in_{/}-\in)\end{array}\right\}$$

$$+\&c. \qquad \&c.$$

The first term of this equation is what was expressed by $\frac{da}{a}$, for if all the periodic quantities be omitted $r = a$, and this equation becomes

$$\frac{da}{a}=-2K-\frac{\left(r-\frac{1}{2}f\right)}{3a^2}+\frac{M^2}{n^2}-\Sigma\frac{m_{/}}{2}a^2\left(\frac{dA_0}{da}\right);$$

for N differs so little from n that $\frac{n^2}{N^2}=1$, without sensible error: this is the permanent change in the radius vector from the disturbing influence.

These are the principal perturbations in the radii vectores of the satellites.

821. Since the squares of the eccentricity are omitted $\sqrt{1-e^2}=1$, and as $m=1$, equation (156) of the longitude becomes

$$dv=\frac{2d(rdr)}{a^2\cdot ndt}-\frac{dr.dr}{a^2\cdot ndt}+3a\iint ndt.dR+2a\int ndt.r\left(\frac{dR}{dr}\right) \qquad \textbf{(259)}$$

since the sign of R is changed.

If the preceding value of $\frac{rdr}{a^2}$ be put in this equation, and also if substitution be made for dR and $r\left(\frac{dR}{dr}\right)$ derived from equation (257), observing that $\frac{1+m_{/}}{a^3}=\frac{1}{a^3}=n^2$, and that M and N differ but little from n, the result will be

$$d\,v = +nt\left\{3K + \frac{\left(r - \frac{1}{2}f\right)}{a^2} - \frac{7}{4}\frac{M^2}{n^2} + \Sigma . m_{/}a^2\left(\frac{dA_0}{da}\right)\right\}$$
$$+\frac{11}{8}\cdot\frac{M^2}{n^2}\cdot\sin 2\left(nt - Mt + \in - E\right)$$
$$+\Sigma\frac{m_{/}n}{n-n_{/}}\left\{\begin{array}{l} +\left\{\frac{n}{n-n_{/}}aA_1 + \frac{2N^2}{\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_{/}}aA_1\right)\right\} \\ \times\sin\left(n_{/}t - nt + \in_{/} - \in\right) \\ +\frac{1}{2}\left\{\frac{n}{n-n_{/}}aA_2 + \frac{2N^2}{4\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_{/}}aA_2\right)\right\} \\ \times\sin 2\left(n_{/}t - nt + \in_{/} - \in\right) \\ +\frac{1}{3}\left\{\frac{n}{n-n_{/}}aA_3 + \frac{2N^2}{9\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_3}{da}\right) + \frac{2n}{n-n_{/}}aA_3\right)\right\} \\ \times\sin 3\left(n_{/}t - nt + \in_{/} - \in\right) \end{array}\right\}$$
$$+\,\&c. \qquad \&c.$$

By article 540 $d\,v$ ought not to contain the mean motion nt, so the first term must be zero, by which the arbitrary constant quantity is determined to be

$$K = -\frac{\left(r - \frac{1}{2}f\right)}{3a^2} + \frac{7}{12}\frac{M^2}{n^2} - \frac{1}{3}\Sigma\, m_{/}a^2\left(\frac{dA_0}{da}\right),$$

whence

$$\frac{d\,a}{a} = \frac{\left(r - \frac{1}{2}f\right)}{3a^2} - \frac{1}{6}\frac{M^2}{n^2} + \frac{1}{6}\Sigma\, m_{/}a^2\left(\frac{dA_0}{da}\right)$$

and

$$N^2 = n^2\left\{1 - 2\frac{\left(r - \frac{1}{2}f\right)}{a^2} - \frac{3}{2}\frac{M^2}{n^2} + \Sigma\, m_{/}a^2\left\{\left(\frac{dA_0}{da}\right) + \frac{1}{2}a\left(\frac{d^2A_0}{da}\right)\right\}\right\}$$

The preceding value of $d\,v$, deprived of its first term, contains all the perturbations in longitude that are independent of the eccentricities; and as the square of s, the tangent of the latitude, is omitted, by article 548 the very small angle $d\,v$ may either be estimated on the orbit of the satellite, or on the fixed plane, since it coincides with its projection. The term depending on the action of the sun corresponds with the variation[3] in the motion of the moon.

822. If the masses of the four satellites be m, m_1, m_2, m_3, the perturbations that m experiences by the action of the other two will be found by changing successively the quantities relative to m_1 into those belonging to m_2 and m_3, and the sum of these will be the action of the three satellites m_1, m_2, and m_3 on m. The perturbations of the others are found by making similar changes.

823. Hereafter the four satellites will be represented m, m_1, m_2, m_3. Where m is the first, or that nearest Jupiter, and m_3 is the fourth and the most distant, all quantities related to them will be accented in the same manner, except it be stated to the contrary.

824. Because $2n_/ = n = N$ nearly,

$$\frac{2m_/ n \cdot N^2}{n-n_/} = m_/ n^2,$$

and the perturbations expressed by

$$\frac{rdr}{a} = +\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_/}\cdot aA_1\right\}.\cos(n_/t - nt + \epsilon_/ - \epsilon)$$
$$+\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\cos 2(n_/t - nt + \epsilon' - \epsilon)$$

[and]

$$dv = +\frac{2m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_/}\cdot aA_1\right\}.\sin(n_/t - nt + \epsilon_/ - \epsilon)$$
$$+\frac{2m_/ n^2}{4(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\sin 2(n_/t - nt + \epsilon_/ - \epsilon)$$

are the greatest to which the three first satellites are liable, on account of the very small divisors arising from the nearly commensurable ratios in the mean motions of these three bodies.

825. The greatest inequality in the first satellite is occasioned by the action of the second, and expressed by

$$\frac{rdr}{a} = +\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\cos 2(n_/t - nt + \epsilon_/ - \epsilon)$$

[and]

$$dv = +\frac{2m_/ n^2}{4(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\sin 2(n_/t - nt + \epsilon_/ - \epsilon)$$

Because the mean motion of the first satellite is nearly double that of the second, $n = 2n_/$, and as $N = n = 2n_/$ nearly, the divisor

$$4\left(n-n_{/}\right)^{2}-N^{2}=\left\{\left(2n-2n_{/}\right)-N\right\}\left\{\left(2n-2n_{/}\right)+N\right\}=2n\cdot\left(2n-2n_{/}-N\right);$$

and if to abridge

$$F=-a^{2}\left(\frac{dA_{2}}{da}\right)-\frac{2n}{n-n_{/}}\cdot aA_{2},$$

the greatest inequalities in the motion of the first satellite are

$$\frac{rdr}{a^{2}}=-\frac{m_{/}n\,.F}{2\left(2n-2n_{/}-N\right)}.\cos 2\left(n_{/}t-nt+\epsilon_{/}-\epsilon\right)$$
$$dv=+\frac{m_{/}n\,.F}{2n-2n_{/}-N}.\sin 2\left(n_{/}t-nt+\epsilon'-\epsilon\right). \qquad \textbf{(260)}$$

826. The principal inequalities in the second satellite arise from the action of the first and third. Those occasioned by the first depend on the terms that have the divisor $\left(n-n_{/}\right)^{2}-N_{/}^{2}$; the quantities having one accent belong to $m_{/}$, the second satellite. Let $A_{1}^{(1.2)}$ be the value of A_{1} when the second satellite is troubled by the first; then if

$$G=-a_{/}^{2}\left(\frac{dA_{1}^{(1.2)}}{da_{/}}\right)+\frac{2n_{/}}{n-n_{/}}\cdot a_{/}A_{1}^{(1.2)},$$

the principal inequalities in the second satellite occasioned by the first are

$$\frac{r_{/}dr_{/}}{d_{/}^{2}}=-\frac{mn_{/}\,.G}{2\left(n-n_{/}-N_{/}\right)}.\cos\left(nt-n_{/}t+\epsilon-\epsilon_{/}\right)$$
$$dv_{/}=+\frac{mn_{/}\,.G}{n-n_{/}-N_{/}}.\sin\left(nt-n_{/}t+\epsilon-\epsilon_{/}\right) \qquad \textbf{(261)}$$

for

$$n=2n_{/},\quad N_{/}=n_{/},$$

and

$$\left(n-n_{/}\right)^{2}-N^{2}=\left\{n_{/}-n-N_{/}\right\}\cdot\left\{n_{/}-n+N_{/}\right\}=2n_{/}\left(n-n_{/}-N_{/}\right)$$

The action of the third satellite on the second is perfectly similar to the action of the second on the first, on account of the ratios $n=2n_{/}$ and $n_{/}=2n_{2}$ in their mean motions; therefore, the inequalities in the motion of the second, occasioned by the action of the third, will be obtained from equations (260), by changing what relates to the first and second into the quantities relative to the second and third. In this case let $A_{2}^{(3.2)}$ be the value of A_{2} and let

$$F' = -a_{/}^{2}\left(\frac{dA_2^{(3.2)}}{da_{/}}\right) - \frac{2n_{/}}{n_{/} - n_2}\cdot a_{/} A_2^{(3.2)}$$

be the value of F, then

$$\begin{aligned} \frac{r_{/} d r_{/}}{a_1^2} &= -\frac{m_2 n_{/} \,.\, F'}{2(n_1 - n_2 - N_{/})}.\cos 2\left(n_{/} t - n_2 t + \epsilon_{/} - \epsilon_2\right) \\ d v_{/} &= +\frac{m_2 n_1 \,.\, F'}{2n_1 - 2n_2 - N_{/}}.\sin 2\left(n_{/} t - n_2 t + \epsilon_{/} - \epsilon_2\right). \end{aligned} \qquad \textbf{(262)}$$

By observation,

$$nt - 3n_{/} t + 2n_2 t + \epsilon - 3\epsilon_{/} + 2\epsilon_2 = 180^\circ,$$

consequently,[4]

$$2\left(n_{/} t - n_2 t + \epsilon_{/} - \epsilon_2\right) = nt - n_{/} t + \epsilon - \epsilon_{/} - 180^\circ;$$

for

$$n = 2n_{/} \qquad n_{/} = 2n_2 \text{ nearly},$$

the two last inequalities may be added to the preceding, since they depend on the same angle; the principal inequalities in the motion of the second satellite from the action of the first and third are therefore[5]

$$\begin{aligned} \frac{r_{/} d r_{/}}{a_1^2} &= -\frac{n_{/}}{2(n - n_{/} - N_{/})}\left\{mG - m_2 F'\right\}.\cos\left(nt - n_{/} t + \epsilon - \epsilon_{/}\right) \\ d v_{/} &= +\frac{n_{/}}{n - n_{/} - N_{/}}\left\{mG - m_2 F'\right\}.\sin\left(nt - n_{/} t + \epsilon - \epsilon_{/}\right). \end{aligned} \qquad \textbf{(263)}$$

In consequence of the ratios in the mean motions these inequalities will never be separated.

827. The action of the second satellite produces inequalities in the theory of the third, analogous to those occasioned by the action of the first on the second; hence, if all the quantities in equations (261) relating to the second and first be changed into those belonging to the third and second, and if $A_{/}^{(2.3)}$ and G' be the values of $A_{/}^{(1.3)}$ and G in this case, so that

$$A_{/}^{(2.3)} = A_{/}^{(1.2)} + \frac{a_2}{a_{/}^2} - \frac{a_1}{a_2^2},$$

and

$$G' = -a_2^2\left(\frac{dA_{/}^{(2.3)}}{da_2}\right) + \frac{2n_2}{n_1 - n_2}\cdot a_2 A_{/}^{(2.3)},$$

the resulting equations for m_2 will be[6]

$$\frac{r_2 d r_2}{a_2^{\;2}} = -\frac{m_1 n_2 G'}{2(n_1 - n_2 - N_2)} . \cos(n_1 t - n_2 t + \epsilon_1 - \epsilon_2)$$
$$d v_2 = +\frac{m_1 n_2 G'}{n_1 - n_2 - N_2} \cdot \sin(n_1 t - n_2 t + \epsilon_1 - \epsilon_2). \qquad \textbf{(264)}$$

These inequalities have only been detected by observation in the motion of the first satellite.

828. G, which is a function of $A_/^{(1.2)}$ may be expressed by a function of $A_/$, for

$$A_/^{(1.2)} = \frac{a_/}{a^2} - \frac{a}{a_/^2} + A_/,$$

whence on account of

$$n = \frac{1}{a^3}, \quad n_/ = \frac{1}{a_/^3};$$

and that $n = 2n_/$ it may be found that

$$G = 2a_/ A_/ - a_/^2 \left(\frac{dA_/}{da_/}\right).$$

SECOND APPROXIMATION

Inequalities depending on the First Powers of the Eccentricities

829. If $a + \dfrac{rd\,r}{a}$ be put for r, equation (255) becomes

$$0 = \frac{d^2 rd\,r}{dt^2} + \frac{rd\,r}{a^3}\left\{1 - \frac{3rd\,r}{a^2}\right\} + 2\int dR + r\left(\frac{dR}{dr}\right)$$

or as

$$\frac{1}{a^3} = n^2 = N^2, \text{ very nearly,}$$

$$0 = \frac{d^2 rd\,r}{dt^2} + N^2 rd\,r\left\{1 - \frac{3rd\,r}{a^2}\right\} + 2\int dR + r\left(\frac{dR}{dr}\right). \qquad \textbf{(265)}$$

If the action of the sun be omitted, the only part of the preceding value of R, that has a sensible effect on the radius vector is

$$R = m_{/}\left\{\begin{array}{l} A_1\cos\left(n_{/}t - nt + \in_{/} - \in\right) + \dfrac{r_{/}dr_{/}}{a_{/}^2}\cdot a_{/}\dfrac{dA_1}{da_{/}}\cos\left(n_{/}t - nt + \in_{/} - \in\right) \\ -2A_1\dfrac{d\left(r_{/}dr_{/}\right)}{a_{/}^2\cdot n_{/}dt}\sin\left(n_{/}t - nt + \in_{/} - \in\right)\end{array}\right\};$$

but these terms are very important, because they serve for the determination of the secular inequalities in the eccentricities and motions of the apsides. With regard to the terms depending on nt, $\int dR = R$, substituting for R, and dividing the whole equation (265) by a^2, it becomes, when $\left(\dfrac{rdr}{a^2}\right)^2$ is omitted,

$$0 = +\frac{d^2 rdr}{a^2dt^2} + N^2\cdot\frac{rdr}{a^2}$$

$$+\Sigma\left\{\begin{array}{l} +m_{/}n^2\dfrac{r_{/}dr_{/}}{a_{/}^2}\left\{2aa_{/}\left(\dfrac{dA_1}{da_{/}}\right) + a^2a_{/}\left(\dfrac{d^2A_1}{dada_{/}}\right)\right\}\cos\left(n_{/}t - nt + \in_{/} - \in\right) \\ -\dfrac{2m_{/}n^2.d\ .r_{/}dr_{/}}{a_{/}^2\cdot n_{/}dt}\left\{2aA_1 + a^2\left(\dfrac{dA_1}{da}\right)\right\}\sin\left(n_{/}t - nt + \in_{/} - \in\right)\end{array}\right\}$$

830. In order to integrate this equation, it may be assumed that

$$\frac{rdr}{a^2} = h\cos\left(nt + \in - gt - \Gamma\right);\quad \frac{r_{/}dr_{/}}{a_{/}^2} = h_{/}\cos\left(n_{/}t + \in_{/} - gt - \Gamma\right),\ \&c.;$$

h and $h_{/}$ are indeterminate coefficients, and $gt + \Gamma$ is the motion of the apsides of the orbits of the satellites.

When these quantities and their differentials are substituted, the square of g neglected, and those terms alone retained that depend on the angle $nt + \in - gt - \Gamma$, a comparison of the coefficients of similar cosines gives

$$0 = h\left\{N^2 + 2ng - n^2\right\} + \Sigma\frac{m_{/}n^2}{2}h_{/}\left\{2aa_{/}\left(\frac{dA_1}{da_{/}}\right) + a^2a_{/}\left(\frac{d^2A_1}{dada_{/}}\right) + 4aA_1 + 2a^2\left(\frac{dA_1}{da}\right)\right\}$$

but by article 458,

$$a\left(\frac{dA_1}{da}\right) + a_{/}\left(\frac{dA_1}{da_{/}}\right) = -A_1;$$

and if the value of N^2 in article 819 be substituted, this coefficient becomes[7]

$$0 = +h\left\{\frac{g}{n} - \frac{\left(r - \tfrac{1}{2}f\right)}{a^2} - \tfrac{3}{4}\frac{M^2}{n^2} + \tfrac{1}{2}\Sigma m_{/}\left\{a^2\left(\frac{dA_0}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2A_0}{da^2}\right)\right\}\right\}$$
$$+\tfrac{1}{2}\Sigma m_{/}h_{/}\left\{aA_1 - a^2\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\}.$$

And is in article 474, if

$$(0.1) = -\frac{m_{/}n}{2}\left\{a^2\left(\frac{dA_0}{da}\right) + \tfrac{1}{2}a^3\left(\frac{d^2A_0}{da^2}\right)\right\};$$
$$\boxed{0.1} = \frac{m_{/}n}{2}\left\{aA_1 - a^2\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\};$$

and if

$$(0) = \frac{\left(r - \tfrac{1}{2}f\right)}{a^2}n; \quad \boxed{0} = \tfrac{3}{4}\frac{M^2}{n},$$

this equation becomes

$$0 = h\left\{g - (0) - \boxed{0} - (0.1)\right\} + \boxed{0.1}\,h_{/}$$

with regard to the first satellite troubled by the second; but the action of m_2 and m_3 produces terms similar to those caused by m_1; and if the same notation be used that was employed for the planets, this equation, when m is troubled by the other three satellites, by the sun, and by the compression of Jupiter, becomes

$$0 = h\left\{g - (0) - \boxed{0} - (0.1) - (0.2) - (0.3)\right\} + \boxed{0.1}\,h_1 + \boxed{0.2}\,h_2 + \boxed{0.3}\,h_3 \qquad \textbf{(266)}$$

A similar equation exists for each satellite, and may be determined from this by changing the quantities relative to m into those relating to m_1 m_2 m_3, and reciprocally; hence, for the others,

$$0 = h_1\left\{g - (1) - \boxed{1} - (1.0) - (1.2) - (1.3)\right\} + \boxed{1.0}\,h + \boxed{1.2}\,h_2 + \boxed{1.3}\,h_3,$$
$$0 = h_2\left\{g - (2) - \boxed{2} - (2.0) - (2.1) - (2.3)\right\} + \boxed{2.0}\,h + \boxed{2.1}\,h_1 + \boxed{2.3}\,h_3, \qquad \textbf{(267)}$$
$$0 = h_3\left\{g - (3) - \boxed{3} - (3.0) - (3.1) - (3.2)\right\} + \boxed{3.0}\,h + \boxed{3.1}\,h_1 + \boxed{3.2}\,h_3.$$

By (484)

$$(0.1)m\sqrt{a} = (1.0)m_{/}\sqrt{a_{/}}, \text{ \&c.}$$

and also

$$\boxed{0.1}\,m\sqrt{a} = \boxed{1.0}\,m_{/}\sqrt{a_{/}}, \text{ \&c.}$$

for any two satellites, so these functions are easily deduced from one another, which saves computation.

These results are perfectly similar to those obtained for the planets, $h\ h_{/}$, &c., correspond to $N\ N'$, &c.

831. It has already been mentioned that the part of the longitude of each satellite depending on the eccentricity consists of four terms, of one that is really the equation of the centre, and of three others arising from the variations in the orbits, chiefly induced by the action of the excess of matter at Jupiter's equator. The coefficients of these sixteen terms are obtained by the aid of the preceding equations, and also of the annual and sidereal motions of the apsides of the orbits. The variations in the radii vectores depend on the same cause, contain the same values of g, and have the same coefficients. $h,\ h_1,\ h_2,\ h_3,$ are the real eccentricities of the four orbits, and if they be eliminated there will result an equation of the fourth degree in g. These four values of g, which will be represented by $g,\ g_1,\ g_2,\ g_3,$ are the annual and sidereal motions of the apsides of the orbits of the four satellites.

832. Let g, the annual and sidereal motion of the first satellite, belong to the first of the preceding equations, and assume $h_1 = x_1 h;\ \ h_2 = x_2 h;\ \ h_3 = x_3 h;$ then the substitution of these in equation (266) will make h vanish, and $x_1,\ x_2,\ x_3,$ will be given in functions of g. Thus h, which may be regarded as the real eccentricity of the orbit of m, is an arbitrary quantity, known by observation. Again, if $g_{/}$ be the value of g in the second of the preceding equations, and if

$$h = x_1^{(1)} h_1, \quad h_2 = x_2^{(1)} h_1, \quad h_3 = x_3^{(1)} h_1,$$

by the substitution of these, h_1 will vanish from the equation in question, $x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},$ will be given in functions of $g_{/}$; and $h_{/}$, the real eccentricity of the orbit of $m_{/}$, is determined by observation. In the same manner, if $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_1^{(3)}, x_2^{(3)}, x_3^{(3)},$ be the quantities corresponding to g_2 and g_3, h_2 and h_3 will be arbitrary constant quantities, which vanish from the two last of equations (267); whence $x_1^{(2)}, x_2^{(2)}, x_3^{(2)},$ and $x_1^{(3)}, x_2^{(3)}, x_3^{(3)},$ will be given in functions of g_2 and g_3.

Thus the coefficients of the sixteen terms of the equations of the centre, corresponding to the four values of g, are $h,\ h_1,\ h_2,\ h_3,\ x_1 h_1,\ x_2 h_1,\ x_3 h_1,\ x_1^{(1)} h_2,$ &c. &c., of which $h,\ h_1,\ h_2,\ h_3,$ are the real eccentricities of the orbits of the four satellites, and are determined by observation: by means of these, and the equations (266) and (267), values of $x,\ x_1,$ &c. will be obtained; and also the four roots of g. Observation shows, however, that h and h_1 are insensible.

833. It was assumed, that

$$\frac{rdr}{a^2} = h\cos\left(nt + \in - gt - \Gamma\right);$$

and as g has four roots, to each of which there are four corresponding values of h, this expression becomes

$$\frac{rdr}{a^2} = +h\cos(nt+\epsilon-gt-\Gamma) + h_1\cos(nt+\epsilon-g_1t-\Gamma_1)$$
$$+h_2\cos(nt+\epsilon-g_2t-\Gamma_2)+h_3\cos(nt+\epsilon-g_3t-\Gamma_3):$$

thus the whole variation in the radius vector of the first satellite depends on h, the eccentricity of its own orbit, on g the motion of its own nodes, and on those of the other three. The corresponding inequalities in the radii vectores of the other three satellites are,

$$\frac{r_1dr_1}{a_1^2} = +x_{/}\cdot h\cdot\cos(n_1t+\epsilon_1-gt-\Gamma) + x_1^{(1)}h_1\cos(n_1t+\epsilon_1-g_1t-\Gamma_1)$$
$$+x_1^{(2)}h_2\cdot\cos(n_1t+\epsilon_1-g_2t-\Gamma_2)+x_1^{(3)}h_3\cos(n_1t+\epsilon_1-g_3t-\Gamma_3)$$
$$\frac{r_2dr_2}{a_2^2} = +x_2\cdot h\cdot\cos(n_2t+\epsilon_2-gt-\Gamma) + x_2^{(1)}h_1\cos(n_2t+\epsilon_2-g_1t-\Gamma_1)$$
$$+x_2^{(2)}h_2\cdot\cos(n_2t+\epsilon_2-g_2t-\Gamma_2)+x_2^{(3)}h_3\cos(n_2t+\epsilon_2-g_3t-\Gamma_3)$$
$$\frac{r_3dr_3}{a_3^2} = +x_3\cdot h\cdot\cos(n_3t+\epsilon_3-gt-\Gamma) + x_3^{(1)}h_1\cos(n_3t+\epsilon_3-g_1t-\Gamma_1)$$
$$+x_3^{(2)}h_2\cdot\cos(n_3t+\epsilon_3-g_2t-\Gamma_2)+x_3^{(3)}h_3\cos(n_3t+\epsilon_3-g_3t-\Gamma_3).$$

These equations contain the perturbations in the radii vectores of the four satellites, depending on the first powers of the eccentricities, and are the complete integrals of the differential equation (265), when applied to each satellite, since they contain the eight arbitrary constant quantities h, h_1, h_2, h_3, Γ, Γ_1, Γ_2, Γ_3, all of which are known by observation. The four last are the mean longitudes of the lower apsides of the orbits of the satellites at the epoch.

834. If the orbits be considered as variable ellipses, *ae* being the eccentricity of the orbit of the first satellite, and v the longitude of its lower apsis, estimated from the origin of the angles,

$$\frac{rdr}{a^2} = -e\cos(nt+\epsilon-v);$$

comparing this with the preceding value of $\frac{rdr}{a^2}$ the result is

$$e\cos v = -h\cos(gt+\Gamma)-h_1\cos(g_1t+\Gamma_1)-\&c.$$
$$e\sin v = -h\sin(gt+\Gamma)-h_1\sin(g_1t+\Gamma_1)-\&c.$$

whence e and v may be obtained; and for the same reasons, e_1, v_1, e_2, v_2, and e_3, v_3.

835. When the squares of the eccentricity are omitted, the elliptical part of the longitude is $v = 2e\sin(nt + \in - v)$ by 392; or representing it by dv for the satellites, where it chiefly arises from the disturbing forces, it gives

$$dv = 2e\cos v \sin(nt + \in) - 2e\sin v \cos(nt + \in);$$

and substituting for $e\cos v$, and $e\sin v$,

$$dv = -2h\sin(nt + \in - gt - \Gamma) - 2h_1\sin(nt + \in - g_1t - \Gamma_1)$$
$$-2h_2\sin(nt + \in - g_2t - \Gamma_2) - 2h_3\sin(nt + \in - g_3t - \Gamma_3),$$

which is the equation of the centre of the first satellite. It appears, that the first term depends on the eccentricity and apsis of its own orbit, the second term arises from the action of the second satellite, and depends on the eccentricity and apsis of the orbit of that body; the other two inequalities arise from the attraction of the third and fourth satellites, and depend on the eccentricities and apsides of their orbits.

The corresponding inequalities in the longitude of the other three satellites are,

$$dv_1 = -2x_1h\sin(n_1t + \in_1 - gt - \Gamma) - 2x_1^{(1)}h_1\sin(n_1t + \in_1 - g_1t - \Gamma_1)$$
$$-2x_1^{(2)}h_2\sin(n_1t + \in_1 - g_2t - \Gamma_2) - 2x_1^{(3)}h_3\sin(n_1t + \in_1 - g_3t - \Gamma_3)$$
$$dv_2 = -2x_2h\sin(n_2t + \in_2 - gt - \Gamma) - 2x_2^{(1)}h_1\sin(n_2t + \in_2 - g_1t - \Gamma_1)$$
$$-2x_2^{(2)}h_2\sin(n_2t + \in_2 - g_2t - \Gamma_2) - 2x_2^{(3)}h_3\sin(n_2t + \in_2 - g_3t - \Gamma_3)$$
$$dv_3 = -2x_3h\sin(n_3t + \in_3 - gt - \Gamma) - 2x_3^{(1)}h_1\sin(n_3t + \in_3 - g_1t - \Gamma_1)$$
$$-2x_3^{(2)}h_2\sin(n_3t + \in_3 - g_2t - \Gamma_2) - 2x_3^{(3)}h_3\sin(n_3t + \in_3 - g_3t - \Gamma_3).$$

These inequalities are very considerable in the motions of the satellites in longitude.

The whole then depends on the resolution of the equations (266) and (267); these, however, are not complete, as several terms arise from the perturbations depending on the squares and products of the disturbing forces.

Action of the Sun depending on the Eccentricities

836. The part of R depending on the action of the sun in the elliptical hypothesis is

$$R = -\tfrac{3}{4}M^2a^2 \cdot \frac{DdD}{D'^2} - \frac{6rdr}{4}M^2\cos(2nt - 2Mt + 2\in - 2E)$$
$$+\tfrac{12}{4}M^2 \cdot \frac{d(rdr)}{ndt}\sin(2nt - 2Mt + 2\in - 2E).$$

But

$$\frac{rdr}{a^2} = h\cos(nt + \epsilon - gt - \Gamma);$$

and

$$\frac{DdD}{D'^2} = H\cos(Mt + E - \Pi),$$

H being the eccentricity of Jupiter's orbit, and Π the longitude the perihelion; hence

$$R = -\tfrac{3}{4}M^2a^2 \cdot H \cdot \cos(Mt + E - \Pi) - \tfrac{9}{4}M^2 \cdot a^2 \cdot h\cos(nt - 2Mt + \epsilon - 2E + gt + \Gamma);$$

and therefore, equation (265) becomes

$$0 = +\frac{d^2rdr}{dt^2} + N^2\frac{rdr}{a^2}\{1 - 3h\cos(nt + \epsilon - gt - \Gamma)\}$$
$$-\tfrac{3}{2}M^2 . H . \cos(Mt + E - \Pi) - 9M^2 . h . \cos(nt - 2Mt + \epsilon - 2E + gt + \Gamma).$$

By article 820, it appears that $\frac{rdr}{a^2}$ contains the terms

$$-\frac{M^2}{n^2} \cdot \cos(2nt - 2Mt + 2\epsilon - 2E);$$

hence

$$-3N^2 \cdot \frac{rdr}{a^2} \cdot h . \cos(nt + \epsilon - gt + \Gamma)$$

contains

$$\tfrac{3}{2}M^2 \cdot h \cdot \cos(nt - 2Mt + \epsilon - 2E + gt + \Gamma),$$

N^2 being very nearly equal to n^2, so that $\frac{N^2}{n^2} = 1$: thus,

$$0 = \frac{d^2rdr}{a^2dt^2} + N^2 \cdot \frac{rdr}{a^2} - \tfrac{15}{2} \cdot M^2 . h . \cos(nt - 2Mt + \epsilon - 2E + gt + \Gamma) - \tfrac{3}{2}M^2 . H . \cos(Mt + E - \Pi),$$

whence by the method of indeterminate coefficients, the integral is

$$\frac{rdr}{a^2} = \frac{15M^2 \cdot h}{4n(2M + N - n - g)}\cos(nt - 2Mt + \epsilon - 2E + gt + \Gamma) + \frac{3M^2 \cdot H}{2n^2} \cdot \cos(Mt + E - \Pi),$$

which is the effect of the sun's action on the radius vector; and if it be substituted in equation (259), the perturbations in longitude depending on the same cause will be

$$dv = -\frac{15M^2 \cdot h}{4n(2M + N - n - g)} \cdot \sin(nt - 2Mt + \epsilon - 2E + gt + \Gamma) - \frac{3M}{n} \cdot H \cdot \sin(Mt + E - \Pi).$$

837. The first term of the second number of this expression corresponds to the evection in the lunar theory, and is only sensible in the motions of the third and fourth satellites; but it is not the only inequality of this kind, for each of the roots $g_1,\ g_2,\ g_3,$ furnishes another. The perturbations corresponding to these for the other satellites are found, by reciprocally changing the quantities relative to one into those relating to the others.

Inequalities depending on the Eccentricities which become sensible in consequence of the Divisors they acquire by double integration

838. It is found by observation, that the mean motion of the first satellite is nearly equal to twice that of the second; and that the mean motion of the second is nearly equal to twice that of the third; or

$$n = 2n_1, \quad \mathrm{n}_1 = 2n_2.$$

In consequence of the squares of these nearly commensurable quantities becoming divisors to the inequalities by a double integration, they have a very sensible effect on the preceding equations in longitude.

839. The only part of equation (259) that has a double integral is $3a\iint ndt \cdot dR;$ and as the divisors in question arise from the angles $nt - 2n_1t,\ n_1t - 2n_2t$ alone, it is easy to see that the part of R containing these angles is,

$$\begin{aligned} R = &+m_1 \frac{r_1 d r_1}{a_1^2} \cdot a_1 \left(\frac{dA_1}{da_1}\right) \cdot \cos(n_1t - nt + \epsilon_1 + \epsilon) \\ &-2m_1' \cdot \frac{d \cdot (r_1 d r_1)}{a_1^2 \cdot n_1 dt} \cdot A_1 \cdot \sin(n_1t - nt + \epsilon_1 - \epsilon) \\ &+m_/ \cdot \frac{rd\,r}{a^2} a \cdot \left(\frac{dA_2}{da}\right) \cdot \cos 2(n_/t - nt + \epsilon_/ - \epsilon) \\ &+4m_/ \frac{d \cdot (rd\,r)}{a^2 \cdot ndt} \cdot A_2 \cdot \sin e(n_/t - nt + \epsilon_/ - \epsilon). \end{aligned}$$

With regard to the action of $m_/$ on m, if $h_/ \cos(n_/t + \epsilon_/ - gt - \Gamma),$ be put instead of $\frac{r_/ d\,r_/}{a_/^2},$ and $h\cos(nt + \epsilon - gt - \Gamma)$ instead of $\frac{rd\,r}{a^2};$ and as by articles 828 and 826

$$G = -a_/^2\left(\frac{dA_1}{da_/}\right) + 2a_/A_1$$
$$F = -a^2\left(\frac{dA_2}{da}\right) - 2aA_2,$$

observing that $n = 2n_/$ nearly, the result will be

$$R = -\frac{m_/}{2a}\cdot\left\{Fh + \frac{a}{a_/}Gh_/\right\}\cdot\cos\left(nt - 2n_/t + \in - 2\in_/ + gt + \Gamma\right),$$

which substituted in $3a\iint ndt\,.\,dR,$ and integrated, gives for the first satellite,

$$d\,v = \frac{-3m_/\,.\,n^2}{2\left(n - 2n_/ + g\right)^2}\cdot\left\{F'h + \frac{a}{a_/}Gh_/\right\}\cdot\sin\left(nt - 2n_/t + \in - 2\in_/ + gt + \Gamma\right).$$

Again, since $n_1 = 2n_2$ nearly, the action of m_2 on $m_/$ produces in $d\,v_/$ an inequality similar to the preceding, which is

$$d\,v_/ = \frac{-3m_2\,.\,n_/^2}{2\left(n_/ - 2n_2 + g\right)^2}\cdot\left\{Fh_/ + \frac{a_1}{a_2}G'h_2\right\}\cdot\sin\left(n_1t - 2n_2t + \in_1 - 2\in_2 + gt + \Gamma\right).$$

An inequality of the same kind, and from the same cause, is produced also in the equation of the centre of $m_/$ by the action of m, for with regard to the inequalities we are now considering, article 574 shows that

$$d\,v_/ = -\frac{m\sqrt{a}}{m_/\sqrt{a_/}}d\,v$$

whence the inequality produced by the action of m on $m_/$ is

$$d\,v_/ = \frac{-3m\,.\,n^2\sqrt{a}}{2\left(n - 2n_/ + g\right)^2\sqrt{a_/}}\cdot\left\{Fh + \frac{a}{a_/}Gh_/\right\}\cdot\sin\left(nt - 2n_/t + \in - 2\in_/ + gt + \Gamma\right).$$

This inequality may be added to the preceding, for

$$nt - 2n_/t + \in - 2\in_/ = n_1t - 2\,n_2t + \in_1 - 2\in_2 + 180^\circ,$$

and as $n = 2n_{/}$ nearly, and $\left(\frac{a}{a_{/}}\right)^2 = \left(\frac{n_{/}}{n}\right)^2$; therefore

$$\frac{n^2\sqrt{a}}{\sqrt{a_{/}}} = 2n_{/}^2 \cdot \frac{a_{/}}{a},$$

and thus the two terms become

$$d\,v_{/} = \frac{3n_{/}^2}{(n-2n_{/}+g)^2} \cdot \left\{ m\left\{Gh_{/} + \frac{a_{/}}{a}Fh\right\} + \frac{m_2}{2}\left\{F'h_{/} + \frac{a_{/}}{a_2}G'h_2\right\}\right\} \cdot \sin\left(nt - 2n_{/}t + \in -2\in_{/} + gt + \Gamma\right).$$

Lastly, the action of m_1 on m_2 produces an inequality in m_2, analogous to that produced by the action of m on $m_{/}$, which is therefore

$$d\,v_2 = \frac{-3m_{/}\,.\,n_2^2}{(n_1 - 2n_2 + g)^2} \cdot \left\{G'h_2 + \frac{a_2}{a_1}F'h_{/}\right\} \cdot \sin\left(nt - 2n_{/}t + \in -2\in_{/} + gt + \Gamma\right).$$

We shall represent the preceding inequalities by

$$d\,v = -Q\sin\left(nt - 2n_{/}t + \in -2\in_{/} + gt + \Gamma\right) \quad \textbf{(268)}$$

$$d\,v_1 = +Q_1\sin\left(nt - 2n_{/}t + \in -2\in_{/} + gt + \Gamma\right) \quad \textbf{(269)}$$

$$d\,v_2 = -Q_2\sin\left(nt - 2n_{/}t + \in -2\in_{/} + gt + \Gamma\right) \quad \textbf{(270)}$$

These inequalities are relative to the root g, but each of the roots g_1, g_2, g_3, give similar inequalities in the motions of the three first satellites.

No such inequality exists in the motion of the fourth satellite, since its mean motion is not nearly commensurable with that of any of the others.

Inequalities depending on the Square of the Disturbing Force

840. On account of the nearly commensurable ratios in the mean motions of the three first satellites the preceding equations must be added as periodic variations to the mean motions, as in the case of Jupiter and Saturn, by means of them several terms are added to equations (266) and (267), which determine the secular variations in the eccentricities and longitudes of the apsides. For if the eccentricities be omitted, and $m = 1$, the equations df, df' in article 433 relative to the planets, become

$$d\left(e\cos v\right) = -andt\left\{2\cos v\left(\frac{dR}{dv}\right) + a\sin v\left(\frac{dR}{dr}\right)\right\},$$

$$d\left(e\sin v\right)=-andt\left\{2\sin v\left(\frac{dR}{dv}\right)-a\cos v\left(\frac{dR}{dr}\right)\right\}.$$

The secular variations with regard to the first satellite will be found by substituting

$$R=-\frac{\left(r-\frac{1}{2}f\right)}{3r^3}+m_1A_2\cos 2\left(v_{/}-v\right)$$

in the first of the preceding equations, and putting $nt+\in+dv$ for v, and a^2+2rdr for r^2; whence

$$\begin{aligned}d\left(e\cos v\right)=&+4andt\cdot m_{/}A_2\sin\left(2v-2v_{/}\right)\cos v\\&-a^2ndt\cdot m_{/}\left(\frac{dA_2}{da}\right)\cos\left(2v-2v_{/}\right)\sin v\\&-ndt\cdot\frac{\left(r-\frac{1}{2}f\right)}{a^2}\cdot\sin\left(nt+\in\right)\\&-ndt\cdot\frac{\left(r-\frac{1}{2}f\right)}{a^2}dv\cos\left(nt+\in\right)\\&+4ndt\cdot\frac{\left(r-\frac{1}{2}f\right)}{a^2}\cdot\frac{rdr}{a^2}\sin\left(nt+\in\right).\end{aligned}$$

Then on1y attending to the terms depending on $nt-2n_{/}t+\in-2\in_{/}$, if the values of $\frac{rdr}{a^2}$ and dv given by (260) be substituted; and as

$$F=-4aA_2-a^2\left(\frac{dA_2}{da}\right),$$

the result will be

$$d\left(e\cos v\right)=-\frac{m_{/}F\,.\,ndt}{2}\cdot\left\{1-\frac{(0)}{2n-2n_{/}-N}\right\}\cdot\sin\left(nt-2n_{/}t+\in-2\in_{/}\right)$$

in which

$$(0)=\frac{\left(r-\frac{1}{2}f\right)}{a^2}n\,.$$

Since the mean longitudes $nt+\in$ and $n_{/}t+\in_{/}$ are variable, these angles must be augmented by the values of dv, $dv_{/}$, in equations (268) and (269), so that

$$nt + \epsilon + Q\sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + gt + \Gamma\right)$$
$$n_{/}t + \epsilon_{/} + Q_{/}\sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + gt + \Gamma\right)$$

must be substituted in the sine of the preceding equation, which becomes, in consequence,

$$d\left(e\cos v\right) = \frac{m_{/}F\,.\,ndt}{4}\cdot\left\{1 - \frac{(0)}{2n - 2n_{/} - N}\right\}\cdot\left(2Q_{/} - Q\right)\cdot\sin\left(gt + \Gamma\right)$$

when the periodic part is omitted. But by article 834,

$$e\cos v = -h\cos\left(gt + \Gamma\right);$$

hence

$$d\left(e\cos v\right) = hg\cdot dt\cdot\sin\left(gt + \Gamma\right),$$

and thus

$$\frac{m_{/}\,.\,Fn}{4}\cdot\left\{1 - \frac{(0)}{2n - 2n_{/} - N}\right\}\cdot\left(2Q_{/} - Q\right)$$

must be subtracted from equation (266).

841. The same analysis applied to $d\left(e_{/}\cos v_{/}\right)$ will determine the increment of the first of equations (267), with regard to the second satellite. But, in this case,

$$R = -\frac{\left(r - \frac{1}{2}f\right)}{3r^{3}} + m_{/}A_{1}^{(1.2)}\cos\left(v - v_{/}\right) + m_{2}A_{2}^{(3.2)}\cos 2\left(v_{/} - v_{2}\right),$$

and equations (269) and (270) must be employed. The result is, that

$$\frac{m_{2}n_{/}}{4}\left\{1 - \frac{(1)}{n - n_{/} - N_{/}}\right\}F'\cdot\left(2Q_{2} - Q_{1}\right) - \frac{mn_{/}}{4}\cdot\left\{\frac{(1)}{n - n_{/} - N_{/}}\right\}G\cdot\left(2Q_{/} - Q\right)$$

must be added to the first of equations (267).

For the same reason

$$\frac{m_{/}n_{2}}{4}\cdot G'\cdot\left(2Q_{2} - Q_{1}\right)\cdot\left\{1 - \frac{(2)}{n_{/} - n_{2} - N_{2}}\right\}$$

must be added to the second of equation (267).

As these quantities only arise from the ratios among the mean motions of the three first satellites, the secular variations of the fourth are not affected by them. In consequence of these additions, equations (266) and (267) become

$$
\begin{aligned}
0 &= +h\left\{g-(0)-\boxed{0}-(0.1)-(0.2)-(0.3)\right\}+\boxed{0.1}h_1+\boxed{0.2}h_2+\boxed{0.3}h_3 \\
&\quad -\frac{m_{/}n}{4}\left\{1-\frac{(0)}{2n-2n_{/}-N_{/}}\right\}F\left(2Q_{/}-Q\right); \\
0 &= +h_1\left\{g-(1)-\boxed{1}-(1.0)-(1.2)-(1.3)\right\}+\boxed{1.0}h+\boxed{1.2}h_2+\boxed{1.3}h_3 \\
&\quad -\frac{mn_{/}}{4}\left\{1-\frac{(1)}{n-n_{/}-N_{/}}\right\}G\left(2Q_{/}-Q\right)+\frac{m_2n_{/}}{4}\left\{1-\frac{(1)}{n-n_{/}-N_{/}}\right\}F'\left(Q_2-Q_1\right); \\
0 &= +h_2\left\{g-(2)-\boxed{2}-(2.0)-(2.1)-(2.3)\right\}+\boxed{2.0}h+\boxed{2.1}h_1+\boxed{2.3}h_2 \\
&\quad +\frac{m_{/}n_2}{4}\left\{1-\frac{(2)}{n_{/}-2n_2-N_2}\right\}G'\left(2Q_2-Q_1\right); \\
0 &= +h_3\left\{g-(3)-\boxed{3}-(3.0)-(3.1)-(3.2)\right\}+\boxed{3.0}h+\boxed{3.1}h_1+\boxed{3.2}h_2 \,.
\end{aligned}
\qquad \textbf{(271)}
$$

842. An inequality which is only sensible in the theory of the second satellite may now be determined; for, by (260),

$$d\,v=\frac{m_{/}nF}{2n-2n_{/}-N}\sin\left(2nt-2n_{/}t+2\in-2\in_{/}\right);$$

or[8]

$$d\,v=\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\left\{\cos\left(nt-2n_{/}t+\in-2\in_{/}\right).\sin\left(nt+\in\right)+\sin\left(nt-2n_{/}t+\in-2\in_{/}\right).\cos\left(nt+\in\right)\right\};$$

but as $v=2e\sin\left(nt+\in-v\right)$, and for the variable ellipse which we are now considering,

$$d\,v=2d\cdot\left(e\cos v\right)\cdot\sin\left(nt+\in\right)-2d\cdot\left(e\sin v\right)\cdot\cos\left(nt+\in\right).$$

By comparing these two values,

$$
\begin{aligned}
2d\left(e\sin v\right)&=-\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\sin\left(nt-2n_{/}t+\in-2\in_{/}\right), \\
2d\left(e\sin v\right)&=+\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\cos\left(nt-2n_{/}t+\in-2\in_{/}\right).
\end{aligned}
$$

But the elliptical expression of v contains the term

$$\tfrac{5}{4}e^2\sin\left(2nt+2\in-2v\right),$$

or

$$\tfrac{5}{4}\left(e^2\cos^2 v-e^2\sin^2 v\right).\sin 2\left(nt+\in\right)-\tfrac{5}{4}.e^2\sin v\ .\cos v\ .\cos 2\left(nt+\in\right).$$

If $e\sin v + d(e\sin v)$, and $e\cos v + d(e\cos v)$ be put for $e\sin v$, and $e\cos v$, it becomes[9]

$$d\,v = \tfrac{5}{4}\left\{(d\cdot e\cos v)^2 - (d\cdot e\sin v)^2\right\}\cdot\sin 2(nt+\epsilon) - \tfrac{5}{4} d\cdot e\cos v\cdot d\cdot e\sin v\cdot\cos 2(nt+\epsilon);$$

and in consequence of the preceding values of $d(e\cos v)$, $d(e\sin v)$, there is the following inequality in the longitude of the first satellite,

$$d\,v = \tfrac{5}{16}\left(\frac{m_/ nF}{2n - 2n_/ - N}\right)^2 \sin 4(nt - n_/ t + \epsilon - \epsilon_/).$$

By the same process the corresponding inequalities in the second and third satellites are found to be

$$d\,v_/ = \tfrac{5}{16}\frac{n_/^2}{(n - n_/ - N_/)^2}\{mG - m_2 F'\}^2 \sin 2(nt - n_/ t + \epsilon - \epsilon_/)$$

$$d\,v_2 = \tfrac{5}{16}\left(\frac{m_/ n_2 G'}{n_/ - n_2 - N_2}\right)^2 \sin 2(n_1 t - n_2 t + \epsilon_/ - \epsilon_2).$$

Librations of the three first Satellites

843. Some very interesting inequalities arising from the equation

$$nt - 3n_1 t + 2n_2 t + \epsilon - 3\epsilon_1 + 2\epsilon_2 = 180^\circ,$$

are found among the terms depending on the squares of the disturbing forces, that affect the whole theory of the satellites, in consequence of the very small divisor $(n - 3n_1 + 2n_2)^2$ which they acquire by double integration. If the orbits be considered as variable ellipses, and if z, z_1, z_2, be the mean longitudes of the three first satellites, it is clear that the terms having the square of $n - 3n_1 + 2n_2$ for divisor, can only be found from

$$d^2 z = 3andt\,.\,\mathrm{d}R$$
$$d^2 z_1 = 3a_1 n_1 d\,t\,.\,\mathrm{d}R_1$$
$$d^2 z_2 = 3a_2 n_2 dt\,.\,\mathrm{d}R_2$$

which are variations in the mean motions by article 439.

844. With regard to the action of m_1 on m, the series R in article 815 only contains the angle $n_{/}t - nt + \epsilon_{/} - \epsilon$ and its multiples, it is evident therefore, that the angle $nt - 3n_1t + 2n_2t$ can only arise from the substitution of the perturbations (262) which depend on the angle $2n_1t - 2n_2t$. By article 814, $d\,v_1$ contains both the elliptical part of the longitude and the perturbations, and if the latter be expressed by[10] $d\,\overline{v}_{/}$ then[11]

$$d\,v_{/} = \frac{2d\left(r_{/}d\,v_{/}\right)}{a_{/}^2 \cdot n_{/}dt} + d\,\overline{v}_{/}$$

and when the square of the eccentricity is omitted $\frac{r_{/}d\,r_{/}}{a_{/}^2}$ becomes $\frac{d\,r_{/}}{a_{/}}$. If then $d\,\overline{v}_{/}$ and $\frac{d\,r_{/}}{a_{/}}$ be put for $\frac{2d\left(r_{/}d\,r_{/}\right)}{a_{/}^2 \cdot n_{/}dt}$ and $\frac{r_{/}d\,r_{/}}{a_{/}^2}$ the part of R required is

$$R = m_{/} \cdot \left(\frac{dA_{/}}{da_{/}}\right) \cdot d\,r_{/} \cdot \cos\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) - m_{/} \cdot d\,\overline{v}_{/} \cdot A_{/} \cdot \sin\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) \cdot ndt,$$

or

$$\mathrm{d}R = m_{/} \cdot A_{/}d\,\overline{v}_{/} \cdot \cos\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) \cdot ndt - m_{/} \cdot \left(\frac{dA_{/}}{da_{/}}\right) \cdot d\,r_{/} \cdot \sin\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) \cdot ndt.$$

for in this case $d\,d\,r_{/}$ and $d\,d\,\overline{v}_{/}$ are zero, since equations (262), or

$$d\,r_{/} = -\frac{m_2 n_{/} a_{/} F'}{2\left(n_1 - n_2 - N_{/}\right)} . \cos\left(2n_{/}t - 2n_2t + 2\epsilon_1 - 2\epsilon_2\right)$$

$$d\,\overline{v}_{/} = +\frac{m_2 n_1 F'}{2n_{/} - 2n_2 - N_{/}} . \sin\left(2n_1t - 2n_2t + 2\epsilon_1 - 2\epsilon_2\right)$$

do not contain the arc nt. If these quantities be substituted in dR, it will be found, in consequence of

$$G = 2a_{/}A_1 - a_{/}^2\left(\frac{dA_1}{da_{/}}\right), \text{ and } n = 2n_{/},$$

that

$$\frac{d^2z}{dt^2} = -\frac{3n^2 m_{/} m_2 F' G}{8\left(n - n_{/} - N_{/}\right)} \frac{a}{a_{/}} \sin\left(nt - 3n_1t + 2n_2t + \epsilon - 3\epsilon_{/} + 2\epsilon_2\right);$$

for as

$$2n_{/} - 2n_2 = n - n_{/} \text{ nearly,}$$

the divisor

$$2n_{/} - 2n_2 - N_{/} = n - n_{/} - N_{/}.$$

The variation in the mean motion of the second satellite consists of two parts; one arising from the action of m, and the other from that of m_2.

The value of R for the first is

$$R = m \cdot A^{(1.2)} \cdot d\overline{v}_{/} \cdot \sin\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right) + m \cdot \left(\frac{dA^{(1.2)}}{da}\right) \cdot dr_{/} \cdot \cos\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right).$$

If the differential of R be taken with regard to $n_{/}t$, making $d\overline{v}_{/}$ and $dr_{/}$ vary, by the substitution of the preceding values of $d\overline{v}_{/}$, $dr_{/}$, and their differentials, it will be found, in consequence of

$$G = 2a_{/}A_{/}^{(1.2)} - a_{/}^{2}\left(\frac{dA_{/}^{(1.2)}}{da_{/}}\right),$$

and $n_2 = \frac{1}{2}n_1$, that the variation in the mean motion of the second satellite from the action of the first must be

$$\frac{3n^3 m \cdot m_2 F' G}{16\left(n - n_{/} - N_{/}\right)} \sin\left(nt - 3n_{/}t + 2n_2t + \epsilon - 3\epsilon_{/} - 2\epsilon_2\right).$$

Again, if

$$\frac{dr_{/}}{a_{/}} = -\frac{mn_{/}G}{2\left(n - n_{/} - N_{/}\right)} \cos\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right),$$

and

$$dv_{/} = +\frac{mn_{/}G}{n - n_{/} - N_{/}} \sin\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right),$$

from article 826, be substituted in the differential of

$$R = m_2 \left\{\left(\frac{dA_2^{(3.2)}}{da}\right) dr_{/} \cos\left(2n_{/}t - 2n_2t + 2\epsilon_1 - \epsilon_2\right) - 2A_2^{(3.2)} \cdot d\overline{v}_{/} \cdot \sin\left(2n_{/}t - 2n_2t + 2\epsilon_1 - 2\epsilon_2\right)\right\},$$

which is the value of R with regard to m_2 and m_1, observing that $n = 2n_{/}$; and, by article 826,

$$F' = -4a_{/}A_2^{(3.2)} - a_{/}^{2}\left(\frac{dA_2^{(3.2)}}{da_{/}}\right),$$

the part of $\frac{d^2 z_{/}}{dt^2}$, arising from the action of m_2 on $m_{/}$, will be found equal to

$$\frac{3m \cdot m_2 n^3}{32\left(n - n_{/} - N_{/}\right)} F'G \sin\left(nt - 3n_{/}t + 2n_2 t + \in - 3\in_1 + 2\in_2\right);$$

and the whole variation in the mean motion of $m_{/}$, from the combined action of m and m_2, is

$$\frac{d^2 z_{/}}{dt^2} = \frac{9mm_2 n^3 F'G}{32\left(n - n_{/} - N_{/}\right)} \sin\left(nt - 3n_{/}t + 2n_2 t + \in - 3\in_1 + 2\in_2\right).$$

With regard to the action of m_1 on m_2

$$R = m_1 \left\{ -2A_2^{(3.2)} . d\bar{v}_{/} \sin 2\left(n_1 t - n_2 t + \in_1 - \in_2\right) + \left(\frac{dA_2^{(3.2)}}{da_{/}}\right) \cdot dr_{/} . \cos 2\left(n_1 t - n_2 t + \in_1 - \in_2\right) \right\}.$$

If the same values of $d\bar{v}_{/}$ and $dr_{/}$ be substituted in the differential of this with regard to $n_2 t$, it will be found that the action of m_1 and m_2 produces the inequality

$$\frac{d^2 z_2}{dt^2} = -\frac{3n^3 mm_{/} F'G}{64\left(n - n_{/} - N_{/}\right)} \cdot \frac{a_2}{a} \sin\left(nt - 3n_{/}t + 2n_2 t + \in - 3\in_1 + 2\in_2\right).$$

845. As

$$\frac{d^2 z}{dt^2} = 3andt . \mathrm{d}R; \quad \frac{d^2 z_{/}}{dt^2} = 3a_{/}n_{/}dt . \mathrm{d}R_{/}; \quad \frac{d^2 z_2}{dt^2} = 3a_2 n_2 dt . \mathrm{d}R_2;$$

by comparing the values of these three quantities in the last article the result is

$$mdR + m_{/}dR_{/} = 0, \text{ and } m_{/}dR_{/} + m_2 dR_2 = 0,$$

which is conformable with what was shown in article 573, with regard to the planets.

846. As the three first satellites move in orbits, they are nearly circular, the error would be very small, in assuming[12]

$$nt + \in, \quad n_{/}t + \in_{/}, \quad n_2 t + \in_2,$$

to be their true longitudes.

The preceding inequalities in the mean motions of the three first satellites are therefore

$$\frac{d^2v}{dt^2} = -\frac{3n^3m_1m_2\frac{a}{a_/}F'G}{8(n-n_/-N_/)}\sin(v-3v_/+2v_2)$$
$$\frac{d^2v_/}{dt^2} = +\frac{9n^3mm_2F'G}{32(n-n_/-N_/)}\sin(v-3v_/+2v_2) \qquad \textbf{(272)}$$
$$\frac{d^2v_2}{dt^2} = -\frac{3n^3mm_/F'G}{64(n-n_/-N_/)}\frac{a_2}{a_/}\sin(v-3v_/+2v_2).$$

847. In order to abridge, let $f = v - 3v_1 + 2v_2$; whence

$$\frac{d^2f}{dt^2} = \frac{d^2v}{dt^2} - 3\frac{d^2v_/}{dt^2} + 2\frac{d^2v_2}{dt^2}.$$

If the preceding values be put in this, and if to abridge,

$$K = -\frac{3nF'G}{8(n-n_/-N_/)}\left\{\frac{a}{a_/}m_1m_2 + \tfrac{9}{4}mm_2 + \frac{a_2}{4a_/}mm_2\right\},$$

the result will be[13]

$$\frac{d^2f}{dt^2} = K.n^2.\sin f.$$

K and n^2 may be assumed to be constant quantities, their variations are so small; hence the integral of this equation is

$$dt = \frac{\pm df}{\sqrt{c - 2Kn^2\cos f}};$$

c is a constant quantity introduced by integration, the different values of which give rise to the three following cases.

848. 1st. If c be greater than $2Kn^2$, without regard to the sign, it must be positive; and the angle $\pm f$ will increase indefinitely, and will become equal to one, two, three, &c., circumferences.

2nd. If K be positive, and c less than $2n^2K$, abstracting from the sign, the radical will be imaginary when $\pm f$ is equal to zero, or to one, two, three, &c. circumferences. The angle f must therefore oscillate about the semi-circumference,[14] since it never can he zero, or equal to a whole circumference, which would make the time an imaginary quantity. Its mean value must consequently be 180°.

3rd. If c be less than $2Kn^2$, and K negative, the radical would be imaginary when the angle $\pm f$ is equal to any odd number of semi-circumferences; the angle f must therefore oscillate about zero, its mean value, since the time cannot be imaginary. However, as it will be shown that K is a positive quantity, the latter case does not exist, so that f must either increase indefinitely, or oscillate about 180°. In order to ascertain which of these is the law of nature, let

$$f = p \pm v ,$$

p being 180° and v any angle whatever; hence

$$dt = \frac{dv}{\sqrt{c + 2Kn^2 \cos v}} . \qquad \textbf{(273)}$$

If the angles $\pm f$ and v increase indefinitely, c is positive, and greater than $2Kn^2$; hence, in the interval between $v = 0$, and its increase to 90°, dt is less than

$$\frac{dv}{n\sqrt{2K}}; \text{ and } t < \frac{v}{n\sqrt{2K}} .$$

Thus the time t that the angle v employs in increasing till it be equal to 90°, will be less than

$$\frac{v}{2n\sqrt{2K}} .$$

This time is less than two years: but from the discovery of the satellites the libration or angle v has always been zero, or extremely small; therefore this angle does not increase indefinitely, it can only oscillate about its mean value of zero.

The second case, then, is what really exists, and the angle

$$v - 3v_1 + 2v_2 ,$$

must oscillate about 180°, which is its mean value.

849. Several important results are given by the equation

$$v - 3v_1 + 2v_2 = p + v .$$

If the insensible part v be omitted,

$$nt - 3n_1 t + 2n_2 t + \epsilon - 3\epsilon_1 + 2\epsilon_2 = p .$$

Whence

$$n - 3n_1 + 2n_2 = 0$$
$$\in - 3\in_1 + 2\in_2 = 180^\circ .$$

These two equations are perfectly confirmed by observation, for Delambre[15] found, from the comparison of a great number of eclipses of the three first satellites, that their mean motions in a hundred Julian years, with regard to the equinox, are

1st Satellite	$7,432,435^\circ.46982$
2nd Satellite	$3,702,713^\circ.231493$
3rd Satellite	$1,837,852^\circ.113582$

whence it appears, that the mean motion of the first, minus three times that of the second, plus twice that of the third, is equal to $9''.0072$, so small a quantity, that it affords an astonishing proof of the accuracy both of the theory and observation. Delambre determined also, from a great number of eclipses, that the epochs of the mean motions of the three first satellites, at midnight, on the first of January 1750, were

$$\in = 15^\circ.02626$$
$$\in' = 310^\circ.44689$$
$$\in'' = 10^\circ.27219 ,$$

whence

$$\in - 3\in_1 + 2\in_2 = 180^\circ\ 1'\ 3'',$$

a result that is less accurate than the preceding; but it will be shown, in treating of the eclipses of the satellites, that it probably arises from errors of observation, depending on the discs of the satellites, which vanish to us before they are quite immersed in the shadow.

850. The same laws exist in the synodic motions of the satellites; for in the equation

$$nt - 3n_1 t + 2n_2 t + \in - 3\in_1 + 2\in_2 = 180^\circ,$$

the angles may be estimated from a moveable axis, since the position of the axis would vanish in this equation: we may therefore suppose that

$$nt + \in, \quad n_1 t + \in_1, \quad n_2 t + \in_2,$$

are the mean synodic longitudes. This has a great influence on the eclipses of the three first satellites, as will appear afterwards.

851. On account of these laws the actions of the first and third satellites on the second are united in one term, given in article 826, which is the great inequality in that body indicated by observations. These inequalities will never be separated.

852. Without the mutual attraction of the satellites the two equations

$$n - 3n_1 + 2n_2 = 0$$
$$\epsilon - 3\epsilon_1 + 2\epsilon_2 = 0$$

would be unconnected. It would have been necessary in the beginning of their motions that their epochs and mean motions had been so arranged as to suit these equations, which is most improbable; and in this case the slightest action from any foreign cause, as the attraction of the planets and comets, would have changed the ratios. But the mutual action of the satellites gives perfect stability to these relations, for, at the origin of the motion, when $t = 0$,

$$\frac{dv}{ndt} - 3\frac{dv_1}{n_1 dt} + 2\frac{dv_2}{n_2 dt} = \pm\sqrt{\frac{c}{n^2} - 2K\cos\left(\epsilon - 2\epsilon_1 + 3\epsilon_2\right)}$$

c being less than $2Kn^2$. It would be sufficient for the accuracy of the preceding results that the first member of this equation had been comprised between the limits

$$+2K\sin\left(\tfrac{1}{2}\epsilon - \tfrac{3}{2}\epsilon_1 + \epsilon_2\right)$$
$$-2K\sin\left(\tfrac{1}{2}\epsilon - \tfrac{3}{2}\epsilon_1 + \epsilon_2\right)$$

at the origin of their motions, and it is sufficient for their stability that no foreign force disturbs it.

853. It appears then, that if the preceding laws among the mean motions of the three first satellites had only been approximate at their origin, their mutual attraction would ultimately have rendered them exact.

854. The angle v is so small, that we may make

$$\cos v = 1 - \tfrac{1}{2}v^2;$$

and if to abridge

$$x^2 = \frac{c + 2Kn^2}{n^2 K},$$

x being arbitrary, on account of the arbitrary constant quantity c that it contains, equation (273) becomes

$$v = x\sin\left(nt\sqrt{K} + A\right),$$

A being a new arbitrary quantity.

855. As the motions of the four satellites in longitude, latitude, and distance, are determined by twelve differential equations of the second order, their integrals must contain twenty-four arbitrary quantities, which are the data of the problem, and are given by observation. Two of these are determined by the equations

$$n - 3n_1 + 2n_2 = 0$$
$$\in - 3\in_1 + 2\in_2 = 180^\circ\ ;$$

they are, however, replaced by x and A, the first determines the extent of the libration, and A marks the time when it is zero: neither are determined, since the inequality v has as yet been insensible.

856. The integrals of the three equations (272) may now be found, for as

$$v - 3v_1 + 2v_2 = p + v = p + x \sin\left(nt\sqrt{K} + A\right),$$
$$\sin\left(v - 3v_1 + 2v_2\right) = \sin\left\{p + x \sin\left(nt\sqrt{K} + A\right)\right\}$$
$$= -x \sin\left(nt\sqrt{K} + A\right);$$

hence the first of equations (272) becomes

$$\frac{d^2v}{dt^2} = \frac{3n^2 m_1 m_2 F'G}{8\left(n - n_1 - N_1\right)} \frac{a}{a_1} x \sin\left(nt\sqrt{K} + A\right),$$

the integral of which is

$$v = \frac{x \sin\left(nt\sqrt{K} + A\right)}{1 + \dfrac{9a_1 m}{4am_1} + \dfrac{a_2 m}{4am_2}}.$$

In the same way

$$v_1 = -\frac{x \sin\left(nt\sqrt{K} + A\right) \cdot \dfrac{3a_/ m}{4am_/}}{1 + \dfrac{9a_1 m}{4am_1} + \dfrac{a_2 m}{4am_2}}$$

$$v_2 = \frac{\dfrac{a_2 m}{8am_2} x \sin\left(nt\sqrt{K} + A\right)}{1 + \dfrac{9a_1 m}{4am_1} + \dfrac{a_2 m}{4am_2}}$$

which are the three equations of the libration. They have hitherto been insensible, but they modify all the inequalities of long periods in the theory of the three first satellites.

857. For example, the inequality

$$v = -\frac{3M}{n} H \sin\left(Mt + E - \Pi\right),$$

gives

$$\frac{d^2v}{dt^2} = +\frac{3M^2}{n} H \sin\left(Mt + E - \Pi\right);$$

But the differential of the first of the equations of libration is

$$\frac{d^2v}{dt^2} = -\frac{Kn^2 \sin\left(v - 3v_1 + 2v_2\right)}{1 + \frac{9a_1 m}{4am_1} + \frac{a_2 m}{4am_2}};$$

or, if to abridge,

$$b = 1 + \frac{9a_1 m}{4am_1} + \frac{a_2 m}{4am_2}$$

$$\frac{d^2v}{dt^2} = -\frac{Kn^2}{b} \cdot \sin\left(v - 3v_1 + 2v_2\right),$$

and adding the two values of $\frac{d^2v}{dt^2}$

$$\frac{d^2v}{dt^2} = -\frac{Kn^2}{b} \sin\left(v - 3v_1 + 2v_2\right) + \frac{3M^3}{n} \cdot H \sin\left(Mt + E - \Pi\right). \qquad \textbf{(274)}$$

To integrate this equation let

$$v = l \sin\left(Mt + E - \Pi\right), \quad v_1 = l_1 \sin\left(Mt + E - \Pi\right), \quad v_2 = l_2 \sin\left(Mt + E - \Pi\right),$$

hence,

$$v - 3v_1 + 2v_2 = \left(l - 3l_1 + 2l_2\right) \cdot \sin\left(Mt + E - \Pi\right),$$

and

$$\frac{d^2v}{dt^2} = \left\{\frac{3M^3 . H}{n} - \frac{Kn^2}{b}\left(l - 3l_1 + 2l_2\right)\right\} \sin\left(Mt + E - \Pi\right);$$

and if

$$l \sin\left(Mt + E - \Pi\right)$$

be put for v,

$$l = -\frac{3M \cdot H}{n} + \frac{Kn^2}{bM^2}\left(l - 3l_1 + 2l_2\right).$$

In the same manner it may be found that

$$l_1 = -\frac{6M \cdot H}{n} - \frac{3a_1 m}{4am_1} \cdot \frac{Kn^2}{bM^2}\left(l - 3l_2 + 2l_2 \right),$$

$$l_2 = -\frac{12M \cdot H}{n} + \frac{3a_2 m}{8am_2} \cdot \frac{Kn^2}{bM^2}\left(l - 3l_1 + 2\, l_2 \right),$$

whence

$$l - 3l_1 + 2l_2 = \frac{9M^3 \cdot H}{n\left(Kn^2 - M^2\right)};$$

so that equation (274) becomes

$$\frac{d^2 v}{dt^2} = +\frac{3M^3}{n}\left\{1 + \frac{3K \cdot n^2}{b\left(Kn^2 - M^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

and

$$d\,v = -\frac{3M}{n}\left\{1 + \frac{3K \cdot n^2}{b\left(Kn^2 - M^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right).$$

The inequalities in the longitude of m_1 and m_2 are found by the same analysis, consequently

$$d\,v = -\frac{3M}{n}\left\{1 + \frac{3K \cdot n^2}{b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

$$d\,v_1 = -\frac{6M}{n}\left\{1 - \frac{9a_1 mK \cdot n^2}{8am_1 b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

$$d\,v_2 = -\frac{12M}{n}\left\{1 + \frac{3a_2 mK \cdot n^2}{32am_2 b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right).$$

This inequality replaces the term depending on the same angle in article 836. It corresponds with the annual equation in the lunar theory, and its period is very great.

858. The variation in the form and position of Jupiter's orbit is the cause of secular inequalities in the mean motions of the satellites, similar to those produced by the variation of the earth's orbit on the moon; hitherto, however, they have been insensible, and will remain so for a long time, with the exception of one depending on the displacement of Jupiter's equator, and that is only perceptible in the motions of the fourth satellite; but these cannot be determined till the equations in latitude have been found.

Notes

[1] Galilei, Galileo, *Sidereus Nuncius or the Sidereal Messenger*, University of Chicago Press, 1989 (see also note 1, *Introduction*).

[2] The closing parenthesis in the argument $\left(\cos\left(U-v\right)+3\cos2\left(U-v\right)\right)$ is omitted in the 1st edition.

[3] This word is capitalized in the 1st edition.

[4] The 1st left hand term inside the parenthesis *reads* $nt_{/}$ in the 1st edition.

[5] The right hand side argument in the 1st equation *reads* $\left(\cos nt-n_{/}t+\in-\in_{/}\right)$ in the 1st edition.

[6] The right hand side argument in the 1st equation *reads* $\left(n_{1}t-n_{2}+\in_{1}-\in_{2}\right)$ in the 1st edition.

[7] The closing parenthesis of the 1st term is omitted in the 1st edition.

[8] The term $\sin\left(nt-2n_{/}t+\in-2\in_{/}\right)$ *reads* $\sin\left(nt-2nt+\in-2\in\right)$ in the 1st edition.

[9] The left hand parenthesis is omitted in the term $\left(d\,.\,e\cos v\right)^{2}$ and *reads* $d\,.\,e\cos v\,)^{2}$ in the 1st edition.

[10] This *reads* $\bar{d}v_{/}$ in the 1st edition.

[11] The right hand parenthesis is omitted in the numerator of the 1st term and *reads* $2d(n_{/}d\,v_{/}$ in the 1st edition.

[12] The third term *reads* $nt_{2}+\in_{2}$ in the 1st edition.

[13] $\sin f$ *reads* $\sin j$ in the 1st edition.

[14] This *reads* semicircumference in the 1st edition.

[15] See note 54, *Preliminary Dissertation*.