BOOK III

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CHAPTER I

LUNAR THEORY

665. THERE is no object within the scope of astronomical observation which affords greater variety of interesting investigation to the inhabitant of the earth, than the various motions of the moon: from these we ascertain the form of the earth, the vicissitudes of the tides, the distance of the sun, and consequently the magnitude of the solar system. These motions which are so obvious, served as a measure of time to all nations, until the advancement of science taught them the advantages of solar time; to these motions the navigator owes that precision of knowledge which guides him with well-grounded confidence through the deep.

Phases of the Moon

666. The phases of the moon depend upon her synodic motion, that is to say, on the excess of her motion above that of the sun. The moon moves round the earth from west to east; in conjunction she is between the sun and the earth; but as her motion is more rapid than that of the sun, she soon separates from him, and is first seen in the evening like a faint crescent, which increases with her distance till in quadrature, or 90° from him, when half of her disc is enlightened: as her elongation increases, her enlightened disc augments till she is in opposition, when it is full moon, the earth being between her and the sun. In describing the other half of her orbit, she decreases by the same degrees, till she comes into conjunction with the sun again. Though the moon receives no light from the sun when in conjunction, she is visible for a few days before and after it, on account of the light reflected from the earth.

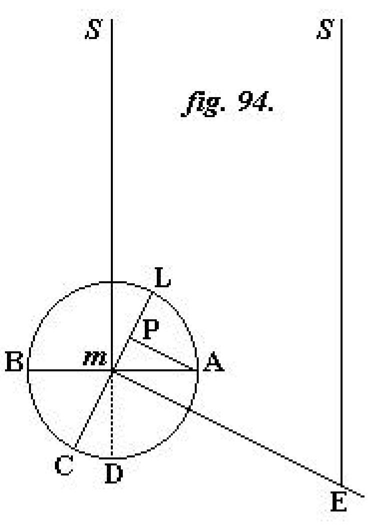


fig. 94.

The law of the variation of the phases of the moon proves her form to be spherical, since they vary as the versed sine of her angular distance from the sun.

If E be the earth, fig. 94, *m* the centre of the moon, supposed to be spherical, and *Sm*, *S*E parallel rays from the sun. Then, if AB be at right angles to the ray *mS*, BLA is the part of the disc that is enlightened by the sun; and CL, being at right angles to *m*E, the part of the moon that is turned to the earth will be CNL; hence the only part of the enlightened disc seen from the earth is LA; or, if it be projected on CL, it is PL, the versed sine of AL. But A*m*L is complement to A*m*N, and is therefore equal to D*m*N, or to *m*ES, the elongation or angular distance of the moon from the sun. When the moon is in quadrature, that is, either 90° or 180° from the sun, a little more than half her disc is enlightened; for when the exact half is visible, the moon is a little nearer to the sun than 90°;

at that instant, which is known by the division between the light and the dark half being a straight line; the lunar radius E*m* fig. 95, is perpendicular to *mS*, the line joining the centres of the sun and moon; hence, in the right-angled triangle E*mS*, the angle E, at the observer, may be measured, and therefore we can determine *S*E, the distance of the sun from the earth, by the solution of a right-angled triangle, when the moon's distance from the earth is known. The difficulty of ascertaining the exact time at which the moon is bisected, renders this method of ascertaining the distance of the sun incorrect. It was employed by Aristarchus of Samos[1] at Alexandria, about two hundred and eighty years before the Christian era,[2] and was the first circumstance that gave any notion of the vast distance and magnitude of the sun.

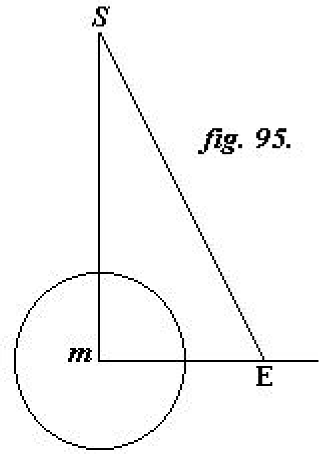


fig. 95.

Mean or Circular Motion of the Moon

667. The mean motion of the moon may be determined by comparing ancient with modern observations. The moon when eclipsed is in opposition, and her place is known from the sun's place, which can be accurately computed back to the earliest ages of antiquity. Three eclipses of the moon observed at Babylon in the years 720 and 719 before the Christian era, are the oldest observations recorded with sufficient precision to be relied on. By comparing these with modern observations, it is found that the mean are described by the moon in one hundred Julian years, or the difference of the mean longitudes of the moon in a century, was $481{,}267^{\circ}.8793$ in the year 1800; it is called the moon's tropical motion, which, omitting 1336 entire circumferences, is $307^{\circ}.8793$; and dividing it by 365.25, the number of days in the Julian year, her diurnal tropical motion is $13^{\circ}.17636$, about thirteen times greater than that of the sun.

668. From the tropical motion of the moon, her periodic revolution, or the time she employs in returning to the same longitude, may be found by simple proportion; for

$$481{,}267^{\circ}.8793 : 360^{\circ} :: 365.25 : 27.321582,$$

the periodic revolution of the moon, or a periodic lunar month.

669. By subtracting $5010''$, or the precession of the equinoxes for a century, from the secular tropical motion of the moon, her sidereal motion in a century is $481{,}266^{\circ}.48763$; or, omitting the whole circumferences, it is $306^{\circ}.48763$; whence, by simple proportion, her sidereal revolution is $27^{d}\ 7^{h}\ 43'\ 11''.5$. These two motions of the moon only differ by the precession of the equinoxes: her sidereal daily motion is, therefore, $13^{\circ}\ 10'\ 35''.034$.

670. The synodic revolution of the moon is her mean motion from conjunction to conjunction, or from opposition to opposition. The mean motion of the moon in a century being $481{,}267^{\circ}.8793$, and that of the sun being $36{,}000^{\circ}.7625$, their difference, $445{,}267^{\circ}.1168$, is the excess of the moon's motion above the sun's in one hundred Julian years; hence her motion through 360° is accomplished in $29^d\ 12^h\ 44'\ 2''.8,$ a lunar month. The lunar month is to the tropical as 19 to 235 nearly, so that 19 solar years are equal to 235 lunar months. The mean motion of the moon is variable, which affects all the preceding results.

671. The apparent diameter of the moon is either measured by a micrometer, or computed from the duration of the occultations of the fixed stars. Its greatest value is thus found to be $2{,}011''.1,$ and the least $1{,}761''.91$. The analogous values in the apparent diameter of the sun are $1{,}955''.6$ and $1{,}890''.96;$ whence the variations in the moon's distance from the earth are much greater than those of the sun; consequently the eccentricity of the lunar orbit is much greater than that of the terrestrial orbit.

672. It appears from observation, that the horizontal parallax of the moon takes all possible values between the limits $1^\circ.0248$ and $0^\circ.8975$ which give 55.9164 and 63.8419 for the least and greatest distances of the moon from the earth; consequently, her mean distance is nearly sixty times the terrestrial radius. The solar parallax shows, that the sun is immensely more distant. Because the lunar parallax is equal to the radius of the terrestrial spheroid divided by the moon's distance from the earth, it is evident that, at the same distance of the moon, the parallax varies with the terrestrial radii; consequently, the variations in the parallax not only prove that the moon moves in an ellipse, having the earth in one of its foci, but that the earth is a spheroid.

Elliptical Motion of the Moon

673. The greatest inequality in the moon's motion is the equation of the centre, which was discovered at a very early period: it is by this quantity alone that the undisturbed elliptical motion of a body differs from its mean or circular motion; it therefore arises entirely from the eccentricity of the orbit, being zero in the apsides, where the elliptical motion is the same with the mean motion, and greatest at the mean distance, or in quadratures, where the two motions differ most. Its maximum is found, by observation, to be $6^\circ\ 17'28''$. This quantity which appears to be invariable, is equal to twice the eccentricity; and if the radius be unity, an arc of

$$3^\circ\ 8'\ 44'' = 0.0549003 = e,$$

the eccentricity of the lunar orbit when the mean distance of the moon from the earth is one.

674. In consequence of the action of the sun, the perigee of the lunar orbit has a direct motion in space. Its mean motion in one hundred Julian years, deduced from a comparison of ancient with modern observations, was $4{,}069^\circ.0395$ in 1800, with regard to the equinoxes, which by simple proportion gives $3{,}231^d.4751$ for its tropical revolution, and $3{,}232^d.5807,$ or a little more than nine years for its sidereal revolution; hence its daily mean motion is $6'41''$. These motions change on account of the secular variation in the motion of the perigee.

675. The anomalistic[3] revolution of the moon is her revolution with regard to her apsides, because the moon moves in the same direction with her perigee; after separating from that point, she only comes to it again by the excess of her velocity. That excess is $477{,}198^{\circ}.69184$ in one hundred Julian years; therefore by simple proportion, the moon's anomalistic[4] year is $27^{d}.5546$.

676. The nodes of the lunar orbit have a retrograde motion, which may be computed from observation, in the same manner with the motion of the perigee. The mean tropical motion of the nodes in 1800 was $1{,}936^{\circ}.940733$, which gives $6{,}788^{d}.54019$ for their tropical revolution, and $6{,}793^{d}.42118$ for their sidereal revolution, or $3' 10''.64$ in a day; hence the moon's daily motion, with regard to her node, is $13^{\circ}\ 13' 45''.534$. The motion of the perigee and nodes arises from the disturbing action of the sun, and depends on the ratio of his mass to that of the earth; this being very great, is the reason why the greater axis and nodes of the lunar orbit move so much more rapidly than those of any other body in the system.

Lunar Inequalities

677. The moon is troubled in her motion by the sun; by her own action on the earth, which changes the relative positions of the bodies, and thus affects her motions; by the direct action of the planets; by their disturbing action on the earth, and by the form of the terrestrial spheroid.

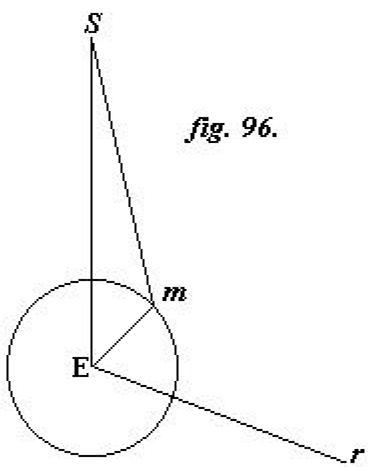


fig. 96.

678. Previous to the analytical investigations, it may perhaps be of use to give some idea of the action of the sun, which is the principle cause of the lunar inequalities.

The moon is attracted by the sun and by the earth at the same time, but her elliptical motion is only troubled by the difference of the actions of the sun on the earth and on herself. Were the sun at an infinite distance, he would act equally and in parallel straight lines, on the earth and moon, and their relative motions would not be troubled by an action common to both; but the distance of the sun although very great, is not infinite. The moon is alternately nearer to the sun and farther from him than the earth; and the straight line *Sm*, fig. 96, which joins the centres of the sun and moon, makes angles more or less acute with *S*E, the radius vector of the earth. Thus the sun acts unequally, and in different directions, on the earth and moon; whence inequalities result in the lunar motions, depending on *m*E*S*, the elongation of the sun and moon, on their distances and the moon's latitude.

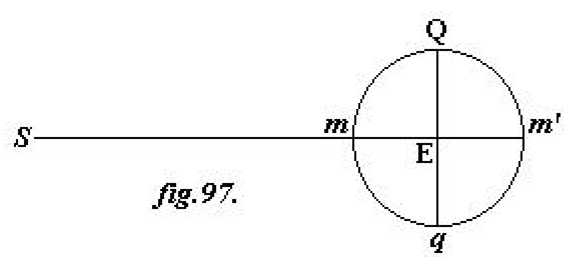
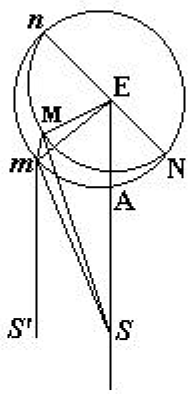


fig. 97.

When the moon is in conjunction at *m*, fig. 97, she is nearer the sun than the earth is; his action is therefore greater on the moon than it is on the earth; the difference of their actions tends to diminish the moon's gravitation to the earth. In opposition at m', the earth is nearer to the sun than the moon is, and therefore the sun attracts the earth more powerfully than he attracts the moon. The difference of these actions tends also to diminish the moon's gravitation to the earth. In quadratures, at Q and *q*, the action of the sun on the moon resolved in the direction of the radius vector QE, tends to augment the gravitation of the moon to the earth; but this increment of gravitation in quadratures is only half of the diminution of gravitation in syzigies; and thus, from the whole action of the sun on the moon in the course of a synodic revolution, there results a mean force directed according to the radius vector of the moon, which diminishes her gravity to the earth, and may be determined as follows:–

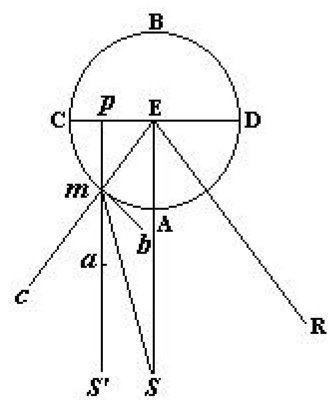
fig. 98.



679. Let M, fig. 98, be the moon in her nearly circular orbit *n*MN; E and *S* the earth and sun in the plane of the ecliptic; *nm*N the moon's orbit projected on the same. Then M*m* is the tangent of the moon's latitude, and E*m* her curtate distance. Let *S*E, E*m*, be represented by r' and *r*, and the angle AE*m* by x, m' being the mass of the sun.

The attraction of the sun on the moon at M is $\frac{m'}{(S\text{M})^2}$. This force may be resolved into three; one in the direction M*m*, which troubles the moon in latitude; another in *m*E, which, being directed towards the centre E, increases the gravity of the moon to the earth, and does not disturb the equable description of areas; and into a third in the direction mS', the excess of which above that by which the sun attracts the earth disturbs the relative position of the moon and earth. The inclination of the lunar orbit is so small that it may be omitted at first, and then the force $\frac{m'}{(Sm)^2}$, fig. 99, is resolved into two, one in the direction mE, which only increases the gravity of the moon, and the other in $N_0, N_1, \ldots$ which disturbs her motion. Let *ma* represent this last force, and suppose it resolved into *mb* and *mc*. The force *mb* accelerates the moon in the quadrants CA and DB, and retards her in the other two; the force *mc* lessens the gravity of the moon.

fig. 99.



680. The analytical expression of these forces is readily found. For the action of the sun on the moon in the direction *Sm*, is $\frac{m'}{(Sm)^2}$, but on account of the great distance of the sun,

$$Sm = \text{SE} - mp = r' - r\cos x, \text{ nearly},$$

hence the action of the sun on the moon in *Sm* is

$$\frac{m'}{\left(r' - r\cos x\right)^2},$$

which, resolved in the direction SE, is

$$\frac{m'}{r'^2} + \frac{m'}{r'^3} 3r\cos x, \text{ nearly}.$$

But the action of the sun on the earth is $\frac{m'}{r'^2}$, and their difference

$$\frac{3m'}{r'^3}\cdot r\cos x$$

is the force *ma*.

Now $\frac{m}{r'^3}\cdot 3r\cos^2 x$, is the force *ma* resolved in *mc*, and

$$\frac{m'}{r'^3}\cdot 3r\sin x\cos x=\frac{m'}{r'^3}\cdot \tfrac{3}{2}r\sin 2x,$$

is the same resolved in *mb*. But the force in *m*E which increases the moon's gravity to the earth, is evidently $\frac{m'r}{r'^3}$; hence the whole force by which the sun increases or diminishes the gravity of the moon to the earth is,

$$\text{force in } m\text{E} - \text{force in } mc, \text{ or } \frac{m'r}{r'^3}\left(1-3\cos^2 . x\right).$$

In syzigy[5] $x=0^\circ$, or 180°, and $\cos^2 x=+1$; thus the action of the sun in conjunction and opposition is $-\frac{2m'r}{r'^3}$. In quadratures $x=90^\circ$, or 270°; hence $\cos x=0$, and the sun's action at these points is $\frac{m'r}{r'^3}$. The mean value of the force $\frac{m'r}{r'^3}\left(1-3\cos^2 x\right)$ for an entire revolution, is the integral of

$$\frac{m'r}{r'^3}\left(1-3\cos^2 x\right)dx=\frac{m'r}{r'^3}\left(1-\tfrac{3}{2}-\tfrac{3}{2}\cos 2x\right)dx,$$

or

$$-\frac{m'r}{r'^3}\left(\tfrac{1}{2}x+\tfrac{3}{4}\sin 2x\right);$$

and when $x=360^\circ$, it becomes $-\frac{m'r}{2r'^3}$, which is the mean disturbing force acting on the moon in the direction of the radius vector.

681. In order to have the ratio of this mean force to the gravity of the moon, we must observe that if E and *m* be the masses of the earth and moon, $\frac{\text{E}+m}{r^2}$ is the force that retains the moon in its orbit, and $\frac{m'}{r'^2}$ is the force that retains the earth in its orbit. But these forces are as

$$\frac{r}{(27.321661)^2} \text{ to } \frac{r'}{(365.25)^2},$$

which are the radii vectores of the moon and earth divided by the squares of their periodic times, whence

$$\frac{mr'}{r'^3} = \frac{1}{179} \cdot \frac{m + \mathrm{E}}{r^2};$$

and thus it appears that the mean action of the sun diminishes the gravity of the moon to the earth by its 358th part, for

$$\frac{mr'}{2r'^3} = \frac{1}{358} \cdot \frac{m + \mathrm{E}}{r^2}.$$

682. In consequence of this diminution of the moon's gravity by its 358th part,[6] she describes her orbit at a greater distance from the earth with a less angular velocity, and in a longer time than if she were urged to the earth by her gravity alone; but as the force is in the direction of the radius vector, the areas are not affected by it; hence, if her radius vector be increased by its 358th part, and her angular velocity diminished by its 179th part, the areas described will be the same as they would have been without that action. The force in the tangent *mb* disturbs the equable description of areas, and that in *m*M troubles the moon in latitude. The true investigation of these forces can only be conducted by an analytical process, which will now be given, without carrying the approximation so far as may be necessary, referring for the complete development of the series, to Damoiseau's profound analysis in the *Memoirs* of the *French Institute* for 1827.[7]

683. The peculiar disturbances to which the moon is liable, and the variety of inferences that may be drawn from them, render her motions better adapted to prove the universal prevalence of the law of gravitation, than those of any other body. The perfect coincidence of theory with observation, shows that analytical formulae not only express all the observed phenomena, but that they may be employed as a means of discovery not less certain than observation itself.

684. Although the motions of the moon be similar to those of a planet, they cannot be determined by the same analysis, on account of the great eccentricity of the lunar orbit, and the immense magnitude of the sun, which make it necessary to carry the approximation at least to the fourth powers of the eccentricities, and to the square of the disturbing force; and although the smallness of the mass of the moon compared with that of the earth, enables us to obtain her perturbations by successive approximations, yet the series converge slowly when the disturbing action of the sun is expressed in functions of the mean longitudes of the sun and moon; and as the facility of analytical investigations, and the fitness of formulae for computation, depend on a skilful choice of co-ordinates, the motions of the moon are first determined in functions of the true longitudes, and then her co-ordinates are obtained by reversion of series in functions of the mean longitudes of the two bodies.

685. The successive approximations are determined by the magnitude of the coefficients. Those terms belong to the first approximation which have for coefficients, either the ratio of the mean motion of the sun to that of the moon, or the eccentricities of the earth and moon, or the inclination of the lunar orbit on the ecliptic. Those terms belong to the second approximation, which have the squares or these quantities as coefficients; those which have their cubes belong to the third, and so on.

The terms having the constant ratio $\frac{a}{a'}=\frac{1}{400}$ of the parallax of the sun to that of the moon for coefficients, are included in the second approximation, and also those depending on the disturbing force of the sun, which is of the order

$$\frac{m'a^3}{a'^3}, \text{ or } m^2;$$

for it has been observed that a permanent change is produced by the disturbing forces in the mean distance: hence if

$$a',\ a,\ n',\ n,\ m',\ m,$$

be the mean distances, mean motions and masses of the sun and moon, and $\bar{a}$ the value of a in the troubled orbit, so that $a=\bar{a}$ when there is no disturbing force, then will

$$\frac{a^2}{\sqrt{\bar{a}}}=\frac{1}{n}, \text{ and as } \frac{m'}{a^3}=n'^2,$$

therefore[8]

$$\frac{m'a^3}{a'^3}\cdot\frac{a}{\bar{a}}=\frac{n'^2}{n^2}=\left(\frac{1}{13.368}\right)^2=0.005595;$$

but the mass of the moon is $m=0.0748013,$ consequently

$$m^2=0.005595,$$

so that

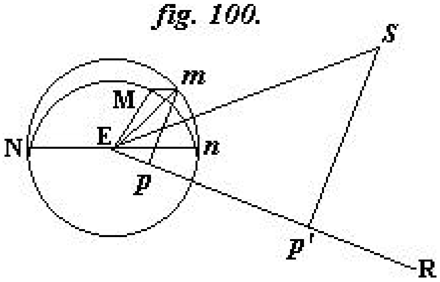
$$\frac{m'a^3}{a'^3}\cdot\frac{a}{\bar{a}}=m^2; \text{ or if } \frac{m'a^3}{a'^3}=\bar{m}^2,$$

then

$$\frac{\bar{m}^2}{\bar{a}}=\frac{m^2}{a}.$$

686. By arranging the series according to the magnitudes of their terms, each approximation may be had separately by taking a certain part and rejecting the rest. This process must be continued till the value of the remainder is so small as to be insensible to observation; but even then it is necessary to ascertain not only that it is so at present, but that it will remain so after the lapse of ages. Besides selecting from the innumerable terms of the series those that have considerable coefficients, it is requisite to examine what values the different terms acquire in the determination of the finite values of the perturbations from their indefinitely small changes, for it has been shown that by integration some of the terms acquire divisors, which increase their values so much that great errors would ensue from omitting them.

Analytical Investigation of the Lunar Inequalities



687. Suppose the motion of the earth to be referred to the sun, and that both sun and moon revolve round the earth assumed to be at rest in E, fig. 100. Let M be the moon in her orbit, *m* her place projected on the plane of the ecliptic, so that E*m* is her curtate distance; and let E*p*, *pm*, M*m*, or x, y, z, be the co-ordinates of the moon, and x', y', z', those of the sun in S, both referred to the centre of the earth, and to the fixed ecliptic at a given epoch.

If m', E, m, be the masses of the sun, the earth, and the moon, the equations of article 347 are

$$0=\frac{d^2x}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dx}\right);$$
$$0=\frac{d^2y}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dy}\right);$$
$$0=\frac{d^2z}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dz}\right).$$

In which $r=\sqrt{x^2+y^2+z^2}$ is the radius vector of the moon,

$$l=\frac{mm'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}};$$

and the element of the time is assumed to be constant in taking the differentials; but if that element be variable, and if [9]

$$R=\frac{(\mathrm{E}+m)}{r}-\frac{m'(x'x+y'y+z'z)}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}+\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}$$

the relative motion of the moon and earth will be determined by the following equations,

$$\frac{d^2x}{dt^2}-\frac{dxd^2t}{dt^3}=\left(\frac{dR}{dx}\right)$$
$$\frac{d^2y}{dt^2}-\frac{dyd^2t}{dt^3}=\left(\frac{dR}{dy}\right)$$
$$\frac{d^2z}{dt^2}-\frac{dzd^2t}{dt^3}=\left(\frac{dR}{dz}\right).$$

688. In very small angles the arc may be taken for its sine; hence the lunar parallax is the radius of the terrestrial spheroid divided by the moon's distance from the earth, and thus the parallax varies inversely as the radius vector. Then if $R^{`}$ be the radius of the earth, and $r$ the radius vector of the moon, the lunar parallax will be $\frac{R^{`}}{r}$, which thus becomes the third co-ordinate of the moon. But if the earth be assumed to be spherical, its radius may be taken equal to unity, and then the lunar parallax will be $\frac{1}{r}$. Therefore let $u=\frac{1}{r}$, [the angle] $\mathrm{RE}m=v$ [fig. 100]; and $m\mathrm{M}=s$, the tangent of the moon's latitude; then

$$r=\sqrt{x^2+y^2+z^2}=\frac{\sqrt{1+ss}}{u},$$

[where]

$$x=\frac{\cos v}{u},\quad y=\frac{\sin v}{u},\quad z=\frac{s}{u}.$$

But in taking the differentials of these, dv must be constant, since dt is assumed to be variable.

689. Let the first of the preceding equations multiplied by $-\sin v$ be added to the second multiplied by $\cos v$; and let the first multiplied by $\cos v$ be added to the second multiplied by $\sin v$; then, if the foregoing values of x, y, z, be substituted, and if to abridge

$$\left(\frac{dR}{dx}\right)\sin v-\left(\frac{dR}{dy}\right)\cos v=p$$
$$\left(\frac{dR}{dx}\right)\cos v+\left(\frac{dR}{dy}\right)\sin v=\Pi,$$

the result will be

$$\frac{d^2v}{dt^2}-\frac{2dvdu}{udt^2}-\frac{dvd^2t}{dt^3}=-pu;$$
$$\frac{d^2u}{u\,.\,dt^2}+\frac{dv^2}{dt^2}-\frac{2du^2}{u^2dt^2}-\frac{dud^2t}{udt^3}=-\Pi u; \qquad \textbf{(206)}$$
$$\frac{d^2s}{dt^2}+\frac{sdv^2}{dt^2}-\frac{dsd^2v}{dvdt^2}=pu\frac{ds}{dv}-\Pi su+u\left(\frac{dR}{dz}\right).$$

The first of these equations multiplied by $\frac{2dv}{u^3}$, and integrated, is

$$\left(\frac{dv}{u^2dt}\right)^2 = h^2 - \int 2p \cdot \frac{dv}{u^3},$$

h^2 being a constant quantity; whence

$$dt = \frac{dv}{u^2\sqrt{h^2 - 2\int \frac{p\,dv}{u^3}}}.$$

The elimination of d^2t between the first and second of equations (206), gives

$$\frac{dud^2v}{u^2dvdt^2} - \frac{d^2u}{u^2dt^2} - \frac{dv^2}{udt^2} = \Pi - \frac{p\,du}{udv};$$

and if dv be assumed to be constant, and substituting for dt its preceding value, it becomes

$$\frac{d^2u}{dv^2} + u = -\frac{\Pi - p\,\frac{du}{udv}}{u^2\left(h^2 - 2\int \frac{p\,dv}{u^3}\right)}.$$

In the same manner the third of equations (206) gives

$$\frac{d^2s}{dv^2} + s = -\frac{\left(\frac{dR}{dz}\right) - \Pi s + \frac{p\,ds}{dv}}{u^2\left(h^2 - 2\int \frac{p\,dv}{u^3}\right)}.$$

Now

$$dR = dx\left(\frac{dR}{dx}\right) + dy\left(\frac{dR}{dy}\right) + dz\left(\frac{dR}{dz}\right),$$

and when substitution is made for dx, dy, dz,

$$dR = -\frac{du}{u^2}\left\{\left(\frac{dR}{dx}\right)\cos v + \left(\frac{dR}{dy}\right)\sin v + \left(\frac{dR}{dz}\right)s\right\}$$
$$-\frac{dv}{u}\left\{\left(\frac{dR}{dx}\right)\sin v - \left(\frac{dR}{dy}\right)\cos v\right\} + \frac{ds}{u}\left(\frac{dR}{dz}\right).$$

But

$$dR = du\left(\frac{dR}{du}\right) + dv\left(\frac{dR}{dv}\right) + ds\left(\frac{dR}{ds}\right);$$

hence, by comparison,

$$\frac{dR}{du} = -\frac{1}{u^2}\left\{\left(\frac{dR}{dx}\right)\cos v + \left(\frac{dR}{dy}\right)\sin v + \left(\frac{dR}{ds}\right)s\right\}$$

$$\frac{dR}{dv} = -\frac{1}{u}\left\{\left(\frac{dR}{dx}\right)\sin v - \left(\frac{dR}{dy}\right)\cos v\right\}$$

$$\frac{dR}{ds} = \frac{1}{u}\left(\frac{dR}{dz}\right).$$

Whence

$$\Pi = -u^2\left(\frac{dR}{du}\right) - su\left(\frac{dR}{ds}\right)$$

$$p = -u\left(\frac{dR}{dv}\right), \text{ and } \frac{dR}{dz} = u\left(\frac{dR}{ds}\right).$$

690. Thus the differential equations which determine the motions of the moon become

$$dt = \frac{dv}{hu^2\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\}^{\frac{1}{2}}};$$

$$0 = \left(\frac{d^2u}{dv^2} + u\right)\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\} + \frac{du}{h^2u^2dv}\left(\frac{dR}{dv}\right) - \frac{1}{h^2}\left(\frac{dR}{du}\right) - \frac{s}{h^2u}\left(\frac{dR}{ds}\right); \qquad \textbf{(207)}$$

$$0 = \left(\frac{d^2s}{dv^2} + s\right)\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\} + \frac{1}{h^2u^2}\cdot\frac{ds}{dv}\left(\frac{dR}{dv}\right) - \frac{s}{h^2u}\left(\frac{dR}{du}\right) - \frac{(1+s^2)}{h^2u^2}\left(\frac{dR}{ds}\right).$$

In the determination of these equations no quantities have been neglected, therefore the influence of such terms as may be omitted in the final result can be fully appreciated.

691. In order to develop[10] the disturbing forces represented by R, the action of the sun alone will be first considered, assuming the masses of the three bodies to be spherical, and $m + \mathrm{E} = 1$. If x', y', z', be the co-ordinates of the sun, and r' its radius vector, then

$$\frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} = \frac{1}{\sqrt{r^2 + r'^2 - 2xx' - 2yy' - 2zz'}}$$

and the second member developed according to the powers of $\frac{1}{r'}$ is

$$\frac{1}{r'}+\frac{xx'+yy'+zz'-\frac{1}{2}r^2}{r'^3}+\frac{3}{2}\frac{\left(xx'+yy'+zz'-\frac{1}{2}r^2\right)^2}{r'^5}+\text{\&c.}$$

692. Since the earth is assumed to be a sphere, its radius may be taken equal to unity, and therefore the solar parallax will be $\frac{1}{r'}$; and if $u'=\frac{1}{r'}$, then $r'=\frac{\sqrt{1+s'^2}}{u'}$. But the ecliptic may be taken for the plane of projection, although it be not fixed; for in its secular motion it carries the orbit of the moon with it, as will be shown afterwards, so that the mean inclination of this orbit on the ecliptic remains constant, and the phenomenon depending on the relative inclination of the two planes are always the same; hence $s'=0$, and therefore $z'=0$, and the co-ordinates of the sun are

$$x'=\frac{\cos v'}{u'}\quad y'=\frac{\sin v'}{u'}.$$

693. Now the distance of the sun from the earth being nearly 400 times greater than that of the moon, $u'=\frac{1}{r'}$ is very small in comparison of $u=\frac{1}{r}$, consequently in the lunar theory u'^5 may be omitted; and if the preceding values of $x, y, z;\ x',\ y',\ \frac{1}{r}$, and $\frac{1}{r'}$, be substituted in R, it becomes

$$R=+\frac{u}{\sqrt{1+s^2}}+m'u'+\frac{m'u'^3}{4u^2}\left\{1+3\cos\left(2v-2v'\right)-2s^2\right\}$$
$$+\frac{m'u'^4}{8u^3}\left\{3\left(1-4s^2\right)\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}. \qquad \textbf{(208)}$$

694. But the quantities $u,\ u',\ v'$, and s, in the elliptical hypothesis, become

$$u+\delta u,\ u'+\delta u',\ v'+\delta v',\ s+\delta s,$$

in the troubled orbit; and as the mass of the sun is so great that the second powers of the disturbing forces must be taken into account, the co-ordinates of the moon must not only contain R but δR.

695. Now

$$\frac{dR}{du}=\frac{1}{\sqrt{1+s^2}}-\frac{m'u'^3}{2u^3}\left\{1+3\cos\left(2v-2v'\right)-2s^2\right\}-\frac{3m'u'^4}{8u^4}\left\{3\left(1-4s^2\right)\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}$$
$$\frac{dR}{dv}=-\frac{3m'u'^3}{2u^2}\sin\left(2v-2v'\right)-\frac{3m'u'^4}{8u^3}\left\{\left(1-4s^2\right)\sin\left(v-v'\right)+5\sin\left(3v-3v'\right)\right\}$$

$$\frac{dR}{ds}=-\frac{su}{\left(1+s^2\right)^{\frac{3}{2}}}-\frac{m'u'^3}{u^2}\cdot s-\frac{3m'u'^4}{u^3}\cdot s\cdot\cos\left(v-v'\right);$$

and if the approximation be only carried to terms of the third order inclusively, the co-ordinates of the moon in her troubled orbit will be

$$\begin{aligned}\frac{d^2u}{dv^2}+u=&+\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}-\frac{m'u'^3}{2h^2u^3}-\frac{3m'u'^3}{2h^2u^3}\cdot\cos\left(2v-2v'\right)+\frac{3m'u'^3}{2h^2u^4}\cdot\frac{du}{dv}\sin\left(2v-2v'\right)\\&-\frac{9}{8}\frac{m'u'^4}{h^2u^4}\cos\left(v-v'\right)+\left(\frac{d^2u}{dv^2}+u\right)\frac{3m'}{h^2}\int\frac{u'^3dv}{u^4}\sin\left(2v-2v'\right)\\&+\left(\frac{d^2u}{dv^2}+u\right)\frac{3m'}{4h^2}\int\frac{u'^4dv}{u^5}\sin\left(v-v'\right)\\&+d\left\{\begin{aligned}&\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}-\frac{m'u'^3}{2h^2u^3}\\&-\frac{3m'u'^3}{2h^2u^3}\cos\left(2v-2v'\right)\\&+\frac{3m'u'^3}{2h^2u^4}\cdot\frac{du}{dv}\sin\left(2v-2v'\right)\\&-\frac{3m'u'^4}{8h^2u^4}\left\{3\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}\\&+\frac{m'u'^4}{8h^2u^5}\cdot\frac{du}{dv}\left\{3\sin\left(v-v'\right)+15\sin\left(3v-3v'\right)\right\}\\&+\frac{3m'}{h^2}\left(\frac{d^2u}{dv^2}+u\right)\left\{\int\frac{u'^3}{u^4}dv\sin\left(2v-2v'\right)+\tfrac{1}{4}\int\frac{u'^4}{u^5}dv\sin\left(v-v'\right)\right\}\\&+\frac{3m'}{h^2}\left\{\frac{d^2u}{dv^2}+u\right\}\left\{\int\frac{u'^4}{4u^5}dv.5\sin\left(3v-3v'\right)\right\}.\end{aligned}\right\}\end{aligned}\qquad\textbf{(209)}$$

$$\begin{aligned}\frac{d^2s}{dv^2}+s=&-\frac{3m'u'^3}{2h^2u^4}s-\frac{3m'u'^3}{2h^2u^4}s\,.\cos\left(2v-2v'\right)+\frac{3m'u'^3}{2h^2u^4}\frac{ds}{dv}\cdot\sin\left(2v-2v'\right)\\&+\frac{3m'u'^4}{2h^2u^5}\left\{11us\cos\left(v-v'\right)+\tfrac{1}{4}\frac{ds}{dv}\sin\left(v-v'\right)\right\}\\&+\frac{3m'}{h^2}\left(\frac{d^2s}{dv^2}+s\right)\cdot\int\frac{u'^3dv}{u^4}\sin\left(2v-2v'\right)\end{aligned}\qquad\textbf{(210)}$$

$$-d\left\{\begin{array}{l}\dfrac{3m'u'^3}{2h^2u^4}s+\dfrac{3m'u'^3}{2h^2u^4}\cos\left(2v-2v'\right)\\ -\dfrac{3m'u'^3}{2h^2u^4}\cdot\dfrac{ds}{dv}\cdot\sin\left(2v-2v'\right)\\ -\dfrac{3m'}{h^2}\left(\dfrac{d^2s}{dv^2}+s\right)\displaystyle\int\dfrac{u'^3dv}{u^4}\sin\left(2v-2v'\right)\end{array}\right\}$$

[and]

$$dt=\frac{dv}{h^2\left(u+du\right)^2\sqrt{1-\dfrac{3m'}{h^2}\displaystyle\int\dfrac{u'^3dv}{u^4}\left\{\sin\left(2v-2v'\right)+\dfrac{u'}{4u}\sin\left(v-v'\right)\right\}}}. \qquad \textbf{(211)}$$

696. These equations contain five unknown quantities, u, v, u', v', and s; but h^2 and v' may be eliminated by their functions in v by integrating the equations (207) when $R=0$, that is, assuming the moon to move without disturbance. By the method already employed, the two last of these equations give

$$s=g\ \sin\left(v-q\right)$$
$$u=\frac{1}{h^2\left(1+g^2\right)}\left\{\sqrt{1+s^2}+e\cos\left(v-v\ \right)\right\}$$

g being the tangent of the inclination of the lunar orbit on the ecliptic, q the longitude of the ascending node, e the eccentricity, and v the longitude of the perigee.

697. In these equations the lunar orbit is assumed to be immovable,[11] but observation shows that the nodes and perigee have a rapid motion in space from the action of the sun; the latter accomplish a revolution in a little more than nine years, so that the lunar ellipse revolves in its own plane in the same direction with the moon's motion; hence if c be such that $1:1-c::v$, the moon's motion in longitude, is to the motion of the apsides, then $v(1-c)$ will be the angle described by the apsis,[12] while the moon describes v. Assuming the instant when the apsis coincided with the axis of x as the origin of the time, then $v-v(1-c)=cv$ will be the moon's true anomaly. In the same manner $(g-1)v$ will represent the retrograde motion of the node, while the moon moves through v. Hence if gv and cv be put for v in the preceding values of s and u, they become

$$s=g\ .\sin\left(gv-q\right)$$
$$u=\frac{1}{h^2\left(1+g^2\right)}\left\{1+\tfrac{1}{4}g^2+e\cos\left(cv-v\ \right)-\tfrac{1}{4}g^2\cos2\left(gv-q\ \right)\right\} \qquad \textbf{(212)}$$

which are the latitude and parallax of the moon in her orbit considered as a revolving ellipse.

This value of u put in $dt = \frac{dv}{h^2u^2}$, which is the first of equations (207), when $R = 0$, gives[13]

$$dt = h^3 dv \left\{ \begin{array}{l} 1 + \frac{3}{2}\left(e^2 + g^2\right) - 2e\left(1 + \frac{3}{2}e^2 + \frac{5}{4}g^2\right)\cos\left(cv - v\right) \\ + \frac{3}{2}e^2\cos\left(2cv - 2v\right) - e^3\cos\left(3cv - 3v\right) + \frac{1}{2}g^2\cos\left(2gv - 2q\right) \\ - \frac{3}{4}eg^2\left\{\cos\left(2gv + cv - v - 2q\right) + \cos\left(2gv - cv + v - 2q\right)\right\} \end{array} \right\}$$

its integral is

$$t = \text{constant} + h^3 \left\{ \begin{array}{l} v\left(1 + \frac{3}{2}e^2 + \frac{3}{2}g^2\right) - \frac{2e}{c}\left(1 + \frac{3}{2}e^2 + \frac{5}{4}g^2\right)\sin\left(cv - v\right) \\ + \frac{3e^2}{4c}\sin\left(2cv - 2v\right) - \frac{e^3}{3c}\sin\left(3cv - 3v\right) + \frac{g^2}{4g}\sin\left(2gv - 2q\right) \\ - \frac{3eg^2}{4\left(2g + c\right)}\sin\left(2gv + cv - v - 2q\right) - \frac{3eg^2}{4\left(2g - c\right)}\sin\left(2gv - cv + v - 2q\right) \end{array} \right\}.$$

698. The coefficients are somewhat modified by the action of the sun. In elliptical motion the coefficient of dv is $a^{\frac{3}{2}}$; a being half the greater axis of the lunar orbit, hence

$$h^3\left(1 + \frac{3}{2}e^2 + \frac{3}{2}g^2\right) = a^{\frac{3}{2}}.$$

699. Again, because $m = 0.0748013$, $c = 1 - \frac{3}{2}m^2 = 0.991548$, $g = 1 + \frac{3}{4}m^2 = 1.00402175$, nearly, therefore c and g may be taken equal to unity in the coefficients of the preceding integral, which becomes, when quantities of the order e^3 are rejected and n put for $a^{-\frac{3}{2}}$,

$$\begin{aligned} nt + \in &= v - 2e\sin\left(cv - v\right) + \frac{3}{4}e^2\sin\left(2cv - 2v\right) \\ &\quad + \frac{1}{4}g^2\sin\left(2gv - 2q\right) - \frac{3}{4}eg^2\sin\left(2gv - cv - 2q + v\right), \end{aligned} \qquad \textbf{(213)}$$

$\in$ being the arbitrary constant quantity.

700. Now, when quantities of the order g^4 are omitted, the coefficient of the second of equations (212) becomes

$$\frac{1}{h^2\left(1 + g^2\right)} = h^{-2}\left(1 - g^2\right);$$

but,

$$h^{-2} = \frac{1}{a}\left(1 + e^2 + g^2 + x\right),$$

x being the remaining part of the development of h^{-2}, and therefore of the fourth order in e and g, consequently

$$\frac{1}{h^2\left(1+g^2\right)}=\frac{1}{a}\left(1+e^2+x\right),$$

and the parallax becomes

$$u=\frac{1}{a}\left\{1+e^2+\tfrac{1}{4}g^2+x+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}$$

the constant part of which is

$$\frac{1}{a}\left(1+e^2+\tfrac{1}{4}g^2+x\right);$$

but as this is modified by the action of the sun, it will be expressed by

$$\frac{1}{\bar{a}}\left(1+e^2+\tfrac{1}{4}g^2+x'\right),$$

so that without that action

$$\frac{1}{a}=\frac{1}{\bar{a}};$$

and when quantities of the fourth order are omitted,

$$u=\frac{1}{\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}. \qquad \textbf{(214)}$$

701. If accented letters are employed for the sun, his parallax and mean longitude will be,

$$u'=\frac{1}{a'}\left\{1+e'^2+e'\left(1+e'^2\right)\cos\left(c'v'-v'\right)\right\} \qquad \textbf{(215)}$$

$$n't+\epsilon'=v'-2e'\sin\left(c'v'-v'\right)+\tfrac{3}{4}e'^2\sin\left(2c'v'-2v'\right). \qquad \textbf{(216)}$$

For $g'=0$ since the sun moves in the plane of the ecliptic, and $g'=1$, $c'=1$, without error in the coefficients.

In order to abridge, let $n't+\epsilon'=v'+f'$, and for the same reason, equation (213) may be expressed by $nt+\epsilon=v+f$. For the elimination of v', suppose the sun and moon to have the same epoch; hence $\epsilon=0$, $\epsilon'=0$, and comparing their mean motions

$$v' = m(v+f) - f', \text{ since } \frac{n'}{n} = m.$$

By the substitution of this in

$$f' = -2e' \sin(c'v' - v') + \tfrac{3}{4}e'^2 \sin(2c'v' - v'),$$

it becomes

$$f' = -2e' \sin\{c'mv - v' + (c'mf - c'f')\} + \tfrac{3}{4}e'^2 \sin\{2c'mv - 2v' + 2(c'mf - c'f')\};$$

or [14]

$$f' = -2e' \sin\{c'mv - v' + c'mf\}\cos c'f' + 2e' \cos\{c'mv - v' + c'mf\}\sin c'f'$$
$$+ \tfrac{3}{4}e'^2 \sin\{2c'mv - 2v' + 2c'mf\}\cos 2c'f' - \tfrac{3}{4}e'^2 \cos\{2c'mv - 2v' + 2c'mf\}\sin 2c'f'.$$

But if

$$c' = 1 \text{ and } \cos c'f' = 1 - \tfrac{1}{2}f'^2 + \&\text{c.}$$
$$\sin c'f' = f' - \tfrac{1}{6}f'^3 + \&\text{c.},$$

then omitting f'^3 the result will be,

$$\begin{aligned} f' = &-2e' \sin(c'mv - v' + c'mf) \\ &+ 2e'f' \cos(c'mv - v' + c'mf) \\ &+ e'f'^2 \sin(c'mv - v' + c'mf) \\ &+ \tfrac{3}{4}e'^2 \sin(2c'mv - 2v' + 2c'mf) \\ &- \tfrac{3}{2}e'^2 \cos(2c'mv - 2v' + 2c'mf) \\ &\quad \&\text{c.} \qquad \&\text{c.} \end{aligned}$$

If substitution be again made for f', and the same process repeated, it will be found, that

$$f' = -e'\left(2 - \tfrac{1}{4}e'^2\right)\sin(c'mv - v') - e'\left(2 - \tfrac{1}{4}e'^2\right)mf \cos(c'mv - v')$$
$$- \tfrac{5}{4}e'^2 \sin(2c'mv - 2v') - \tfrac{5}{2}m'e'^2 f \cos(2c'mv - 2v').$$

If this value of f' be put in $v' = m(v+f) - f'$ the value of f restored, and the products of the sines and cosines reduced to the sines of the sums and differences of the arcs, when e^3 is rejected, the result will be

$$\begin{aligned} v' = &+mv - 2me\sin(cv - v) \\ &+ \tfrac{3}{4}e^2 m\sin(2cv - 2v) \\ &+ \tfrac{1}{4}mg^2\sin(2gv - 2q) \\ &- \tfrac{3}{4}meg^2\sin(2gv - cv + v - 2q) \\ &+ 2e'\left(1 - \tfrac{1}{8}e'^2\right)\sin(c'mv - v') \\ &- 2mee'\sin(cv + c'mv - v - v') \\ &- 2mee'\sin(cv - c'mv - v + v') \\ &+ \tfrac{5}{4}e'^2\sin(2c'mv - 2v'). \end{aligned} \qquad \textbf{(217)}$$

702. If this value of v' be expressed by $v' = mv + y$, and substituted in equation (215) it becomes

$$u' = \frac{1}{a'}\left\{1 + e'^2 + e'\left(1 + e'^2\right)\cos(c'mv - v' + c'y)\right\}.$$

It will readily appear by the same process, when all powers of the eccentricities above the second are rejected, that

$$u' = \frac{1}{a'}\left\{\begin{array}{l} 1 + e'\left(1 - \tfrac{1}{8}e'^2\right)\cos(c'mv - v') + e'^2\cos(2c'mv - 2v') \\ + mee'\cos(cv - c'mv - v + v') - mee'\cos(cv + cmv - v - v') \end{array}\right\}. \qquad \textbf{(218)}$$

703. By the same substitution,

$$\cos(v - v') = \cos(v - m'v)\cos y + \sin(v - mv)\sin y;$$

but

$$\cos y = 1 - \tfrac{1}{2}y^2 + \&c., \quad \sin y = y - \tfrac{1}{6}y^3 + \&c.$$

hence

$$\cos(v - v') = \cos(v - mv) + y\sin(v - mv) - \tfrac{1}{2}y^2\cos(v - mv) - \tfrac{1}{6}y^3\sin(v - mv) + \&c.;$$

and

$$\begin{aligned} \cos(v - v') = &+\cos(v - mv) \\ &- me\cos(v - mv - cv + v) \\ &+ me\cos(v - mv + cv - v) \\ &+ \tfrac{3}{8}me^2\cos(2cv - v + mv - 2v) \\ &- \tfrac{3}{8}me^2\cos(2cv + v - mv - 2v) \\ &+ \tfrac{1}{8}mg^2\cos(2gv - v + mv - 2q) \end{aligned} \qquad \textbf{(219)}$$

$$-\tfrac{1}{8}mg^2\cos\left(2gv+v-mv-2q\right)$$
$$-\tfrac{3}{8}meg^2\cos\left(v-mv-2gv+cv-v+2q\right)$$
$$+\tfrac{3}{8}meg^2\cos\left(v-mv+2gv-cv+v-2q\right)$$
$$+e'\left(1-\tfrac{1}{8}e'^2\right)\cos\left(v-mv-c'mv+v'\right)$$
$$-e'\left(1-\tfrac{1}{8}e'^2\right)\cos\left(v-mv+c'mv-v'\right)$$
$$+\qquad\text{\&c.}\qquad\text{\&c.}$$

Thus the series expressing $\cos\left(v-v'\right)$ may extend to any powers of the disturbing force and eccentricities.

704. Now

$$\cos\left(2v-2v'\right)=+\cos\left(2v-2mv\right)$$
$$+2y\sin\left(2v-2mv\right)$$
$$-2y^2\cos\left(2v-2mv\right)$$
$$-\tfrac{4}{3}y^3\sin\left(2v-2mv\right)$$
$$\text{\&c.}\qquad\text{\&c.}$$

which shows that $\cos\left(2v-2v'\right)$ may be readily obtained from the development of $\cos\left(v-v'\right)$ by putting $2v$ for v, and $2y$ for y; and the same for any cosine, as $\cos i\left(v-v'\right)$.

705. Again, if $90^\circ+v$ be put for v, $\cos\left(v-mv\right)$ becomes

$$\cos\left\{\left(v+90^\circ\right)\left(1-m\right)\right\}=-\sin\left(v-mv\right);$$

hence also the expansion of $\sin\left(v-v'\right)$ may be obtained from the expression (219), and generally the development of $\sin i\left(v-v'\right)$ may be derived from that of $\cos i\left(v-v'\right)$.

Thus all the quantities in the equations of the moon's motions in article 695 are determined, except the variation $d\,u$, $d\,u'$, $d\,v'$, and $d\,s$.

706. It is evident from the value of $\dfrac{d^2u}{dv^2}+u$ in equation (209), that u is a function of the cosines of all the angles contained in the products of the developments[15] of u, u', $\cos\left(v-v'\right)$, $\cos\left(2v-2v'\right)$, &c.; and $d\,u$, being the part of u arising from the disturbing action of the sun, must be a function of the same quantities: hence if A_0, A_1, A_2, &c. be indeterminate coefficients, it may be assumed, that

$$\begin{aligned} a\delta u = &+A_0 \cdot \cos(2v-2mv) \\ &+A_1 e \cdot \cos(2v-2mv-cv+v) \\ &+A_2 e \cdot \cos(2v-2mv+cv-v) \\ &+A_3 e' \cdot \cos(2v-2mv+c'mv-v') \\ &+A_4 e' \cdot \cos(2v-2mv-c'mv+v') \\ &+A_5 e' \cdot \cos(c'mv-v') \\ &+A_6 ee' \cdot \cos(2v-2mv-cv+c'mv+v-v') \\ &+A_7 ee' \cdot \cos(2v-2mv-cv-c'mv+v+v') \\ &+A_8 ee' \cdot \cos(cv+c'mv-v-v') \\ &+A_9 ee' \cdot \cos(cv-c'mv-v+v') \\ &+A_{10} e^2 \cdot \cos(2cv-2v) \\ &+A_{11} e^2 \cdot \cos(2cv-2v+2mv-2v) \\ &+A_{12} g^2 \cdot \cos(2gv-2q) \\ &+A_{13} g^2 \cdot \cos(2gv-2v+2mv-2q) \\ &+A_{14} e'^2 \cdot \cos(2c'mv-2v') \\ &+A_{15} eg^2 \cdot \cos(2gv-cv-2q+v) \\ &+A_{16} eg^2 \cdot \cos(2v-2mv-2gv+cv+2q-v) \\ &+A_{17} \frac{a}{a'} \cdot \cos(v-mv) \\ &+A_{18} \frac{a}{a'} e' \cdot \cos(v-mv+c'mv-v') \\ &+A_{19} \frac{a}{a'} e' \cdot \cos(v-mv-c'mv+v') \\ &+A_{20} \frac{a}{a'} \cdot \cos(3v-3mv). \end{aligned} \tag{220}$$

The term depending on $\cos(cv-v)$ which arises from the disturbing action of the sun is omitted, because it has already been included in the value of u.

707. It is evident from equation (210) that δs, the variation of the tangent of the latitude, can only have the form

$$\begin{aligned} d\,s = &+B_0 \cdot g \,\sin\left(2v - 2mv - gv + q\right) \\ &+B_1 \cdot g \,\sin\left(2v - 2mv + gv - q\right) \\ &+B_2 \cdot eg \,\sin\left(gv + cv - q - v\right) \\ &+B_3 \cdot eg \,\sin\left(gv - cv - q + v\right) \\ &+B_4 \cdot eg \,\sin\left(2v - 2mv - gv + cv + q - v\right) \\ &+B_5 \cdot eg \,\sin\left(2v - 2mv + gv - cv - q + v\right) \\ &+B_6 \cdot eg \,\sin\left(2v - 2mv - gv - cv + q + v\right) \\ &+B_7 \cdot e'g \,\sin\left(gv + c'mv - q - v'\right) \\ &+B_8 \cdot e'g \,\sin\left(gv - c'mv - q + v'\right) \\ &+B_9 \cdot e'g \,\sin\left(2v - 2mv - gv + c'mv + q - v'\right) \\ &+B_{10} \cdot e'g \,\sin\left(2v - 2mv + gv - c'mv - q + v'\right) \\ &+B_{11} \cdot e^2 g \,\sin\left(2cv - gv - 2v + q\right) \\ &+B_{12} \cdot e^2 g \,\sin\left(2v - 2mv - 2cv + gv + 2v - q\right) \\ &+B_{13} \cdot e^2 g \,\sin\left(2cv + gv - 2v + 2mv - 2v - q\right) \\ &+B_{14} \cdot \frac{a}{a'} g \,\sin\left(gv - v + mv - q\right) \\ &+B_{15} \cdot \frac{a}{a'} g \,\sin\left(gv + v - mv - q\right), \end{aligned} \qquad \textbf{(221)}$$

B_0, B_1, &c. being indeterminate coefficients.

708. The variation in the longitude of the earth from the action of the planets troubles the motion of the moon. Equation (216), when $d\,(nt + \in)$ is put for $d\,v$, gives[16]

$$d\,v' = md\,(nt + \in)\left\{1 + 2e'\cos\left(c'mv - v'\right) - \tfrac{5}{2}e'^2\cos\left(2c'mv - 2v'\right)\right\}. \qquad \textbf{(222)}$$

But $d\,v$ or $d\,(nt + \in)$, arising from the disturbing force, is entirely independent of equation (213), which belongs to the elliptical motion only; and from equation (211) it appears that if C_6, C_9, &c. be indeterminate coefficients,

$$\begin{aligned} d\,(nt + \in) = &+C_6 \sin\left(2v - 2mv\right) \\ &+C_9 e' \sin\left(2v - 2mv + c'mv - v'\right) \\ &+C_{10} e' \sin\left(2v - 2mv - c'mv + v'\right) \\ &+ \quad \&\text{c.} \qquad \&\text{c.} \end{aligned} \qquad \textbf{(223)}$$

By this value, equation (222) becomes

$$d\,v' = +m\left\{C_6 + C_9 e'^2 + C_{10} e'^2\right\}\sin\left(2v - 2mv\right) \qquad \textbf{(224)}$$
$$+ \qquad \&c. \qquad \&c.$$

709. But the longitude of the earth is troubled by the action of the moon as well as by that of the planets, and thus the moon indirectly troubles her own motions. In the theory of the earth it is found that the action of the moon occasions the inequality

$$d\,v' = m\frac{r}{r'}\sin\left(v - v'\right)$$

in the earth's longitude, and thus the whole variation of v' is

$$d\,v' = +m\left\{C_6 + C_9 e'^2 + C_{10} e'^2\right\}\sin\left(2v - 2mv\right) + m\frac{u'}{u}\sin\left(v - v'\right); \qquad \textbf{(225)}$$

where m is the ratio of the mass of the moon to the sum of the masses of the earth and moon.

710. The parallax of the moon is troubled by both these causes, but that arising from the action of the planets may be omitted at present. The moon's attraction produces the inequality

$$d\,r' = m\,r\cos\left(v - v'\right)$$

in the radius vector of the earth, and consequently the variation

$$d\,u' = -\frac{mu'^2}{u}\cos\left(v - v'\right) \qquad \textbf{(226)}$$

in the solar parallax.

711. Lastly, $\frac{du}{dv}$ is obtained from equation (214).

712. Thus every quantity in the equation of article 695 are determined, and by their substitution, the co-ordinates of the moon will be obtained in her troubled orbit in functions of her true longitude.

The Parallax

713. The substitution of the given quantities in the differential equation (209) of the parallax is extremely simple, though tedious. The first term

$$-\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}=-\frac{1}{h^2}\left(1-\tfrac{3}{2}s^2\right)$$

when the higher powers of s^2 are omitted; putting

$$\frac{1}{\bar{a}}\left(1+e^2+g^2+x\right) \text{ for } h^{-2}$$

and

$$\tfrac{1}{2}g^2-\tfrac{1}{2}g^2\cos\left(2gv-2q\right) \text{ for } s^2$$

becomes

$$-\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}=-\frac{1}{\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+x+\tfrac{3}{4}g^2\left(1+e^2-\tfrac{1}{4}g^2\right)\cos\left(2gv-2q\right)\right\}.$$

Again,

$$u'^3=\frac{1}{a'^3}\left\{1+\tfrac{3}{2}e'^2+3e'\cos\left(c'mv-v'\right)+\text{\&c.}\right\}$$

$$u^{-3}=a^3\left\{1-\tfrac{3}{4}g^2-3e\cos\left(cv-v\right)+\text{\&c.}\right\};$$

and as by article 685,

$$\frac{m'a^3}{a'^3}=\bar{m}^2$$

$$\frac{m'u^3}{2h^2u^3}=\frac{\bar{m}^2}{2\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+\tfrac{3}{2}e'^2-3e\left(1+\tfrac{1}{2}e^2+\tfrac{3}{2}e'^2\right)\cos\left(cv-v\right)+\text{\&c.}\right\}$$

In this and all the other terms, x is omitted, being of the fourth order in e and g.

714. Terms of the form $\frac{9m'u'^4}{8h^2u^4}\cos\left(v-v'\right)$ become

$$+\frac{9\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}\left(1+2e^2+2e'^2\right)\cos\left(v-mv\right)$$

$$+\frac{9\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}e'\cos\left(v-mv+c'mv-v'\right)$$

$$+\frac{27\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}e'\cos\left(v-mv-c'mv+v'\right);$$

and, by comparing their coefficients with observation, serve for the determination of $\frac{a}{a'}$, the ratio of the parallax of the sun to that of the moon; but as it is a very small quantity, any error would be sensible, and on that account the approximation must extend to quantities of the fifth order inclusively with regard to the angle $v - v'$; but in every other case, it will only be carried to quantities of the third order.

715. Attending to these circumstances, and observing that in the variation of $\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}$ the square of ds must be included, so that

$$d\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}} = -\frac{3s\,ds}{h^2} + \frac{3}{2\bar{a}}ds^2$$

and as

$$\frac{\bar{m}^2}{\bar{a}} = \frac{m^2}{a},$$

it will readily be found, that

$$
\begin{aligned}
0 = \frac{d^2u}{dv^2} &+ u \qquad\qquad \textbf{(227)}\\
&- b_0\\
&- b_1 e\cos\left(cv - v\right)\\
&+ b_2 \cos\left(2v - 2mv\right)\\
&+ b_3 e\cos\left(2v - 2mv - cv + v\right)\\
&- b_4 e\cos\left(2v - 2mv + cv - v\right)\\
&- b_5 e'\cos\left(2v - 2mv + c'mv - v'\right)\\
&+ b_6 e'\cos\left(2v - 2mv - c'mv + v'\right)\\
&+ b_7 e'\cos\left(c'mv - v'\right)\\
&+ b_8 ee'\cos\left(2v - 2mv - cv + c'mv + v - v'\right)\\
&- b_9 ee'\cos\left(2v - 2mv - cv - c'mv + v + v'\right)\\
&- b_{10} ee'\cos\left(cv + c'mv - v - v'\right)\\
&- b_{11} ee'\cos\left(cv - c'mv - v + v'\right)\\
&+ b_{12} e^2\cos\left(2cv - 2v\right)\\
&+ b_{13} e^2\cos\left(2cv - 2v + 2mv - 2v\right)\\
&- b_{14} g^2\cos\left(2gv - 2q\right)\\
&+ b_{15} g^2\cos\left(2gv - 2v + 2mv - 2q\right)\\
&+ b_{16} e'^2\cos\left(2c'mv - 2v'\right)
\end{aligned}
$$

$$-b_{17} e g^{2} \cos\left(2gv - cv - 2q + v\right)$$
$$-b_{18} e g^{2} \cos\left(2v - 2mv - 2gv + cv + 2q - v\right)$$
$$+b_{19} \frac{a}{a'} \cos\left(v - mv\right)$$
$$+b_{20} e' \frac{a}{a'} \cos\left(v - mv + c'mv - v'\right)$$
$$+b_{21} e' \frac{a}{a'} \cos\left(v - mv - c'mv + v'\right).$$

716. The coefficients being[17]

$$b_0 = +\frac{1}{\overline{a}}\left\{1 + e^2 + \tfrac{1}{4}g^{2} + x\right\} - \frac{\overline{m}^2}{2\overline{a}}\left\{1 + e^2 + \tfrac{1}{4}g^{2} + \tfrac{3}{2}e'^2\right\}$$
$$+\frac{3\overline{m}^2}{4\overline{a}}\left(4 - 3m - m^2\right) A_0\left(1 - \tfrac{5}{2}e'^2\right) - \frac{3}{4\overline{a}} B_0^{\,2} g^{2}$$

$$b_1 = \frac{3m^2}{4a}\left\{\begin{array}{l} +2 + e^2 + 3e'^2 - 2\left(B_2 + B_3\right)\dfrac{g^{2}}{m^2} + \left(1 + 2m - c\right) A_0\left(1 - \tfrac{5}{2}e'^2\right) \\ -4\left\{1 + 2m + \left(4\left(1 - m^2\right) - 1\right)\left(\dfrac{1+m}{2 - 2m - c} + \dfrac{1-m}{2 - 2m + c}\right)\right\} \times A_0\left(1 - \tfrac{5}{2}e'^2\right) \\ +\dfrac{1}{1-m}\left\{\left(1 + 6m + c\right)\left(1 + m\right) + 7 + \left(2 - 2m - c\right)^2\right\} A_1\left(1 - \tfrac{5}{2}e'^2\right) \\ -\tfrac{1}{2}\left(9 + m + c\right) A_6 . e'^2 + \tfrac{7}{2}\left(9 + 3m + c\right) A_7 e'^2 + 3\left(A_8 + A_9\right). e'^2 \end{array}\right\}$$

$$b_2 = \frac{3m^2}{4a}\left\{1 + \left(1 + 2m\right)e^2 + \tfrac{1}{4}g^{2} - \tfrac{5}{2}e'^2 + \frac{1}{1-m}\left(1 + 3e^2 + \tfrac{1}{4}g^{2} - \tfrac{5}{2}e'^2\right) - A_0 - \left(B_0 - B_1\right)\frac{g^{2}}{m^2}\right\}$$

$$b_3 = \frac{3m^2}{a}\left\{\begin{array}{l} -\tfrac{1}{4}\left(3 + 4m\right)\left(1 + \tfrac{1}{2}e^2 - \tfrac{5}{2}e'^2\right) + \dfrac{1 - c^2}{4\left(1 - m\right)} \\ -\dfrac{2\left(1 + m\right)}{2 - 2m - c}\left(1 + \tfrac{7}{4}e^2 - \tfrac{5}{2}e'^2\right) - \tfrac{1}{2}\left(A_1 - 2A_0\right) + \tfrac{1}{2}\left(B_5 - B_6\right)\dfrac{g^{2}}{m^2} \end{array}\right\}$$

$$b_4 = \frac{3m^2}{4a}\left\{3 + c - 4m + \frac{8\left(1 - m\right)}{2 - 2m + c} + 2A_2\right\}$$

$$b_5 = \frac{3m^2}{4a}\left\{\frac{4 - m}{2 - m} + 2B_9\frac{g^{2}}{m^2} + 2A_3\right\}$$

$$b_6 = \frac{3m^2}{4a}\left\{\frac{7\left(4 - 3m\right)}{2 - 3m} - 2B_{10}\frac{g^{2}}{m^2} - 2A_4\right\}$$

$$b_7 = +\frac{3m^2}{4a}\left\{\begin{array}{l} +1+e^2+\tfrac{1}{4}g^2+\tfrac{9}{8}e'^2+\left(B_7+B_8\right)\dfrac{g^2}{m^2}-\tfrac{3}{2}\left(1+2m\right)A_0 \\ -\dfrac{2(1-2m)(3-2m)(3-m)}{(2-3m)(2-m)}A_0-2A_3-\left(2-3m\right)A_4 \\ +\left(B_9+B_{10}\right)B_0\dfrac{g^2}{m^2}-A_5-11C_6-2C_9+2C_{10} \end{array}\right\}$$

$$+\frac{6m'}{a}\left\{4A_0+A_3-A_4-10A_1e^2+\tfrac{5}{2}\left(A_7-A_6\right)e^2\right\}$$

$$b_8 = \frac{3m^2}{4a}\left\{\frac{3+2m-c}{4}+\frac{(2+m)}{2-m-c}-\tfrac{3}{2}A_1-A_6-\left(\frac{3+m-c}{2}+\frac{4}{2-m-c}\right)A_9\right\}$$

$$b_9 = \frac{3m^2}{4a}\left\{\frac{7(3+6m-c)}{4}+\frac{7(2+3m)}{2-3m-c}+\tfrac{3}{2}A_1+A_7+\left(\frac{3-m-c}{2}+\frac{4}{2-3m-c}\right)A_8\right\}$$

$$b_{10} = \frac{3m^2}{4a}\left\{\frac{3+2m}{2}-\left(\frac{1+2m+c}{4}+\frac{2}{c+m}\right)A_1+A_8+\left(\frac{1+3m+c}{2}+\frac{4}{c+m}\right)A_7\right\}$$

$$b_{11} = \frac{3m^2}{4a}\left\{\frac{3-2m}{2}+A_9+7\left(\frac{1+2m+c}{4}+\frac{2}{c-m}\right)A_1+\left(\frac{1+m+c}{2}+\frac{4}{c-m}\right)A_6\right\}$$

$$b_{12} = \frac{3m^2}{4a}\left\{1-B_{11}\cdot\frac{g^2}{m^2}-A_{10}\right\}$$

$$b_{13} = \frac{3m^2}{4a}\left\{\frac{2+11m+8m^2}{2}-\frac{\left(10+19m+8m^2\right)}{2c-2+2m}+4A_1+\frac{\left\{8A_{10}+10A_1^2\right\}}{2c-2+2m}-2A_{11}\right\}$$

$$b_{14} = \frac{3}{4a}\left\{1+e^2-\tfrac{1}{4}g^2-\tfrac{1}{2}m^2+2m^2A_{12}\right\}$$

$$b_{15} = \frac{3m^2}{4a}\left\{\frac{1+2m-2g}{4}+\frac{\left(4g^2-1\right)}{4(1-m)}-\frac{(2+m)}{2g-2+2m}+\frac{2B_0}{m^2}-2A_{13}+\frac{8A_{12}}{2g-2+2m}\right\}$$

$$b_{16} = \frac{3m^2}{4a}\left(\tfrac{3}{2}-A_{14}\right)$$

$$b_{17} = \frac{3m^2}{4a}\left\{\tfrac{1}{2}+\frac{B_3}{m^2}+\frac{(1+c-2g-10m)}{4}A_1-\left(10+5m\right)A_{13}+\left(5+m\right)A_{16}-\frac{1}{m^2}B_0B_5+A_{15}\right\}$$

$$b_{18} = \frac{3m^2}{4a}\left\{1+2m+\frac{(5+m)}{1-2m}-\frac{3(1-m)}{3-2m}+2A_{16}-\frac{2}{m^2}B_4+\frac{10}{1-2m}A_{15}\right\}$$

$$b_{19} = \frac{m^2}{a}\left\{\begin{array}{l} +\left\{\tfrac{9}{8}(1-2m)\left(1+2e^2+2e'^2\right)+\dfrac{3(1-2m)\left(1+\tfrac{9}{2}e^2+2e'^2\right)}{4(1-m)}+\dfrac{3(1+m)}{2(1-m)}\right\}\times A_{18}e'^2 \\ -\dfrac{\left(36+21m-15m^2\right)}{4(1-m)}A_{17}-\dfrac{(57-38m)}{4(1-m)}A_0+\tfrac{3}{2}\left(B_{14}+B_{15}\right)\dfrac{g^2}{m^2} \end{array}\right\}$$

$$b_{20} = \frac{3m^2}{4a}\left\{\frac{5(1-2m)}{4} - A_{18} + \frac{(4+m)}{4}A_{17} - (5+m)A_{19}\right\}$$

$$b_{21} = \frac{3m^2}{2a(1-2m)}\left\{\frac{15-18m}{4}(1-2m) - \frac{76-33m}{m}A_{17} - 5A_{18} - (1-2m)A_{19}\right\}.$$

717. The integral of the preceding equation is evidently

$$u = \frac{1}{a}\left\{\begin{matrix} +1+e^2+\tfrac{1}{4}g^2+x+e(1+e^2)\cos(cv-v) \\ -\tfrac{1}{4}g^2\left(1+e^2-\tfrac{1}{4}g^2\right)\cos(2gv-2q) \end{matrix}\right\} + du. \qquad \textbf{(228)}$$

Where du is given by equation (220).

718. In order to find values of the indeterminate coefficients A_0, A_1, &c., this value of u must be substituted in equation (227); but to determine the unknown quantity c, both e and v must vary in the term $e(1+e^2)\cos(cv-v)$, which expresses the motion of the perigee. Hence, when $\dfrac{d\frac{e}{a}}{dv^2}$ is omitted, a comparison of the coefficients of corresponding sines and cosines gives

$$\textbf{(229)}$$

$$0 = 1+e^2+\tfrac{1}{4}g^2+x-ab_0$$

$$0 = 1-\left(c-\frac{dv}{dv}\right)^2 - \frac{ab_1}{1+e^2}$$

$$0 = \frac{e(1+e^2)}{a}\cdot\frac{d^2v}{dv^2} - 2\left(c-\frac{dv}{dv}\right)\frac{d\cdot e\frac{(1+e^2)}{a}}{dv}$$

$$0 = A_0\left(1-4(1-m)^2\right)+ab_2$$

$$0 = A_1\left(1-(2-2m-c)^2\right)+ab_3$$

$$0 = A_2\left(1-(2-2m+c)^2\right)-ab_4$$

$$0 = A_3\left(1-(2-m)^2\right)-ab_5$$

$$0 = A_4\left(1-(2-3m)^2\right)+ab_6$$

$$0 = A_5\left(1 - m^2\right) + ab_7$$

$$0 = A_6\left(1 - \left(2 - m - c\right)^2\right) + ab_8$$

$$0 = A_7\left(1 - \left(2 - 3m - c\right)^2\right) - ab_9$$

$$0 = A_8\left(1 - \left(c + m\right)^2\right) - ab_{10}$$

$$0 = A_9\left(1 - \left(c - m\right)^2\right) - ab_{11}$$

$$0 = A_{10}\left(1 - 4c^2\right) + ab_{12}$$

$$0 = A_{11}\left(1 - \left(2c - 2 + 2m\right)^2\right) + ab_{13}$$

$$0 = A_{12}\left(1 - 4g^2\right) + ab_{14}$$

$$0 = A_{13}\left(1 - \left(2g - 2 + 2m\right)^2\right) + ab_{15}$$

$$0 = A_{14}\left(1 - 4m^2\right) + ab_{16}$$

$$0 = A_{15}\left(1 - \left(2g - c\right)^2\right) - ab_{17}$$

$$0 = A_{16}\left(1 - \left(2 - 2m - 2g + c\right)^2\right) - ab_{18}$$

$$0 = A_{17}\left(1 - \left(1 - m\right)^2\right) + ab_{19}$$

$$0 = ab_{20}$$

$$0 = A_{19}\left(1 - \left(1 - 2m\right)^2\right) + ab_{21}$$

$$0 = A_{20}\left(1 - \left(3 - 3m\right)^2\right).$$

719. The secular inequalities in the form of the lunar orbit are derived from the three first of these equations; from the rest are obtained values of the indeterminate coefficients A_0, A_1, &c. &c. It is evident that these coefficients will be more correct, the farther the approximation is carried in the development of equation (209).

Secular Inequalities in the Lunar Orbit

720. When the action of the sun is omitted, by article 685, $\frac{1}{a} = \frac{1}{\bar{a}}$; and x, being of the fourth order, may be omitted: hence $1 + e^2 + \frac{1}{4}g^2 - ab_0 = 0$ becomes

$$\frac{1}{a} = \frac{1}{\bar{a}} - \frac{\bar{m}^2}{2\bar{a}}\left(1 + \tfrac{3}{2}e'^2\right) + \frac{3\bar{m}^2}{4\bar{a}}\left(1 - \tfrac{5}{2}e'^2\right)\left(4 - 3m - m^2\right)A_0 - \frac{3}{4\bar{a}}B_0^2 g^2. \qquad \textbf{(230)}$$

Since a is the mean distance of the moon from the earth, or half the greater axis of the lunar orbit, the constant part of the moon's parallax is proportional to $\frac{1}{a}$. But the action of the planets produces a secular variation in e', the eccentricity of the terrestrial orbit, without affecting $2a'$, the greater axis. The preceding value of $\frac{1}{a}$ must therefore be subject to a secular inequality, in consequence of the variation of the term $-\frac{3\bar{m}^2}{4\bar{a}}e'^2$; but this variation will always be insensible.

721. The motion of the perigee may be obtained from the second of equations (229), put under the form

$$1-\left(c-\frac{d\varpi}{dv}\right)^2-p-p'e'^2=0\,;$$

for since b_1 is a function of e'^2, the quantity $\frac{ab_1}{1+e^2}$ may be expressed by $p+p'e'^2$.

If $\frac{d\varpi}{dv}$ be omitted, $c=\sqrt{1-p-p'.e'^2}$, so that c varies in consequence of e'^2. Now

$$\frac{d\varpi}{dv}=c-\sqrt{1-p}+\frac{p'e'^2}{2\sqrt{1-p}}\,,$$

the integral of which is

$$\varpi=cv-v\sqrt{1-p}+\frac{p'}{2\sqrt{1-p}}\int e'^2dv+\in\,;$$

for e'^2 is variable, and p, p' may be regarded as constant, without sensible error, as appears from the value of b_1, and $\in$ is a constant quantity, introduced by integration; hence

$$\cos\left(cv-\varpi\right)=\cos\left\{v\sqrt{1-p}-\frac{p'}{2\sqrt{1-p}}\int e'^2dv-\in\right\}. \qquad \textbf{(231)}$$

722. Thus, from theory, we learn that the perigee has a motion equal to

$$\left(1-\sqrt{1-p}\right)v+\frac{p'}{2\sqrt{1-p}}\int e'^2dv\,,$$

which is confirmed by observation; but this motion is subject to a secular inequality, expressed by

$$\frac{p'}{2\sqrt{1-p}}\int e'^2 dv\,, \qquad \textbf{(232)}$$

on account of the variation in e'^2, the eccentricity of the earth's orbit.

In consequence of the preceding value of c, v is equal to the constant quantity $\in$, together with the secular equation of the motion of the perigee.

723. The eccentricity of the moon's orbit is affected by a secular variation similar to that in the parallax, and proportional to $\frac{d\nu}{dv}$, but as the variations of the latter are only sensible in the integral $\int \frac{d\nu}{dv} dv$, the eccentricity of the lunar orbit may be regarded as constant.

Latitude of the Moon

724. The development[18] of the parallax will greatly assist in that of the latitude, as most of the terms differ only in being multiplied by s, its variation, or its differentials; and by substitution of the requisite quantities in equation (210), it will readily be found, when all the powers of the eccentricities and inclination above the cubes are omitted, that

$$\begin{aligned}
0 = &+\frac{d^2 s}{dv^2} + s \\
&+ a_0 g \sin(gv - q) \\
&- a_1 g \sin(2v - 2mv - gv + q) \\
&+ a_2 g \sin(2v - 2mv + gv - q) \\
&+ a_3 eg \sin(gv + cv - q - \varpi) \\
&+ a_4 eg \sin(gv - cv - q + \varpi) \\
&+ a_5 eg \sin(2v - 2mv - gv + cv + q - \varpi) \\
&+ a_6 eg \sin(2v - 2mv + gv - cv - q + \varpi) \\
&+ a_7 eg \sin(2v - 2mv - gv - cv + q + \varpi) \\
&+ a_8 e'g \sin(gv + c'mv - q - \varpi') \\
&+ a_9 e'g \sin(gv - c'mv - q + \varpi') \\
&+ a_{10} e'g \sin(2v - 2mv - gv + c'mv + q - \varpi') \\
&+ a_{11} e'g \sin(2v - 2mv - gv - c'mv + q + \varpi') \\
&+ a_{12} e^2 g \sin(2cv - gv - 2q + \varpi) \\
&+ a_{13} e^2 g \sin(2v - 2mv - 2cv + gv + 2\varpi - q)
\end{aligned} \qquad \textbf{(233)}$$

$$+a_{14}e^2 g \sin\left(2cv+gv-2v+2mv-2v-q\right)$$

$$+a_{15}\frac{a}{a'}g \sin\left(gv-v+mv-q\right)$$

$$+a_{16}\frac{a}{a'}g \sin\left(gv+v-mv-q\right).$$

725. The coefficients in consequence of $\frac{\overline{m}^2}{\overline{a}}=\frac{m^2}{a}$ being

$$a_0=\frac{3m^2}{2}\left\{\begin{aligned}&+1+2e^2-\tfrac{1}{4}g^2+\tfrac{3}{2}e'^2\\&-\tfrac{1}{2}\left(1-\tfrac{5}{2}e'^2\right)\left(\frac{(3-2m-g)(g+m)}{1-m}B_0+4A_0\right)\\&-\tfrac{7}{2}(3-3m-g)B_{10}e'^2+\tfrac{1}{4}(3-m-g)B_9e'^2+\tfrac{3}{2}(B_7+B_8)e'^2\end{aligned}\right\}$$

$$a_1=\frac{3m^2}{4}\left\{\begin{aligned}&+(1+g)\left(1+2e^2-\tfrac{1}{4}(2+m)g^2-\tfrac{5}{2}e'^2\right)\\&+\frac{(1-g^2)}{1-m}-4A_0+10A_1e^2-2B_0\end{aligned}\right\}$$

$$a_2=\frac{3m^2}{2}\left\{\frac{1-g}{2}+B_1\right\}$$

$$a_3=\frac{3m^2}{2}\left\{B_2-2+(1-m)(3-2m-g)B_0\right\}$$

$$a_4=\frac{3m^2}{2}\left\{B_3-2-2A_1+(1+m)(3-2m-g)B_0\right\}$$

$$a_5=\frac{3m^2}{2}\left\{(1+g)(1-m)-2B_0+B_4\right\}$$

$$a_6=\frac{3m^2}{2}\left\{(g-1)(1+m)+B_5-2A_1\right\}$$

$$a_7=\frac{3m^2}{2}\left\{(1+g)(1+m)+B_6+2A_1-2B_0\right\}$$

$$a_8=\frac{3m^2}{4}\left\{3+2B_7+\tfrac{1}{2}(3-2m-g)B_0-(3-3m-g)B_{10}\right\}$$

$$a_9=\frac{3m^2}{4}\left\{3+2B_8-\tfrac{7}{2}(3-2m-g)B_0-(3-m-g)B_9\right\}$$

$$a_{10}=\frac{3m^2}{4}\left\{\frac{1+g}{2}+2B_9+3B_0-(1+g-m)B_8\right\}$$

$$a_{11}=\frac{3m^2}{4}\left\{2B_{10}-\tfrac{7}{2}(1+g)+3B_0-(1+g+m)B_7\right\}$$

$$a_{12} = \frac{3m^2}{4}\left\{\begin{array}{l} +2B_{11} - 5 - 10A_1 + 4A_{11} - (3-2m-2c+2g)B_{12} \\ +(10+19m+8m^2)B_0\left(\dfrac{3-3m-g}{4} + \dfrac{(2-2m-g)^2 - 1}{2(2c+2m-2)}\right)\end{array}\right\}$$

$$a_{13} = \frac{3m^2}{4}\left\{2B_{12} + (1-g)\tfrac{1}{4}(10+19m+8m^2) + 10A_1 - 4A_{11} - 2B_{11}\right\}$$

$$a_{14} = \frac{3m^2}{4}\left\{\tfrac{1}{2}(10+19m+8m^2) + 2B_{13} + 10A_1 - 4A_{11} - 5B_0\right\}$$

$$a_{15} = \frac{3m^2}{4}\left\{3 + 2B_{14}\right\}$$

$$a_{16} = \frac{3m^2}{4}\left\{\tfrac{5}{2} + 2B_{15}\right\}.$$

726. The integral of the differential equation of the latitude is

$$s = g\sin(gv - q) + \delta s;\ \delta s \text{ is given in (221).} \qquad \textbf{(234)}$$

If this quantity be substituted in equation (233) instead of *s*, a comparison of the coefficients of like sines and cosines will furnish a sufficient number of equations; whence the indeterminate coefficients B_0, B_1, &c. will be known, but in order to find a value of the unknown quantity *g*, both q and g must vary in the terms $g\sin(gv-q)$ in taking the differentials of *s*. Attending to these circumstances, it will readily be found that,[19]

$$\textbf{(234)}$$

$$2\frac{dg}{dv}\left(g - \frac{dq}{dv}\right) - g\frac{d^2q}{dv^2} = 0$$

$$g - g\left(g - \frac{dq}{dv}\right)^2 + \frac{d^2g}{dv^2} + g\,a_0 = 0$$

$$B_0\left(1 - (2-2m-g)^2\right) - a_1 = 0$$

$$B_1\left(1 - (2-2m+g)^2\right) + a_2 = 0$$

$$B_2\left(1 - (g+c)^2\right) + a_3 = 0$$

$$B_3\left(1 - (g-c)^2\right) + a_4 = 0$$

$$B_4\left(1 - (2-2m+c-g)^2\right) + a_5 = 0$$

$$B_5\left(1 - (2-2m-c+g)^2\right) + a_6 = 0$$

$$B_6\left(1 - (2-2m-c-g)^2\right) + a_7 = 0$$

$$B_7\left(1-(g+m)^2\right)+a_8=0$$
$$B_8\left(1-(g-m)^2\right)+a_9=0$$
$$B_9\left(1-(2-m-g)^2\right)+a_{10}=0$$
$$B_{10}\left(1-(2-3m-g)^2\right)+a_{11}=0$$
$$B_{11}\left(1-(2c-g)^2\right)+a_{12}=0$$
$$B_{12}\left(1-(2-2m-2c+g)^2\right)+a_{13}=0$$
$$B_{13}\left(1-(2c+g-2+2m)^2\right)+a_{14}=0$$
$$B_{14}\left(1-(g-1+m)^2\right)+a_{15}=0$$
$$B_{15}\left(1-(g+1-m)^2\right)+a_{16}=0.$$

The two first of these equations will give the secular variations in the nodes and inclination of the orbit, the rest serve for the determination of the coefficients B_0, B_1, &c.

Secular Inequalities in the Position of the Lunar Orbit

727. The coefficient a_0 may be represented by $q+q'e'^2$, then the second of the equations in the last article becomes

$$\frac{d^2g}{dv^2}+g\left(1-\left(g-\frac{dq}{dv}\right)^2\right)+g\left(q+q'e'^2\right)=0\,;$$

q' is a function of A_0 and B_0; and as these are functions of $1-\frac{5}{2}e'^2$, therefore $q'e'^2$ may be omitted, as well as $\frac{d^2g}{dv^2}$, which is insensible, and neglecting $\frac{dq}{dv}$ in the first instance,

$$g=\sqrt{1-q-q'e'^2}\,,$$

so that g varies with e'^2. But

$$\frac{dq}{dv}=g-\sqrt{1+q}-\frac{q'}{2\sqrt{1+q}}e'^2\,;$$

and as q and q' may be regarded as constant, the integral is

$$q = gv - v\sqrt{1+q} - \frac{q'}{2\sqrt{1+q}}\int e'^2 dv + a\ ,$$

a being a constant quantity introduced by integration; hence

$$\sin\left(gv - q\right) = \sin\left\{v\sqrt{1+q} + \frac{q'}{2\sqrt{1+q}}\int e'^2 dv - a\right\}, \qquad \textbf{(235)}$$

which shows the nodes of the lunar orbit to have a retrograde motion on the true ecliptic equal to

$$\left(\sqrt{1+q} - 1\right)v + \frac{q'}{2\sqrt{1+q}}\int e'^2 dv\ ,$$

which accords with observation. This motion is not uniform, but is affected by a secular inequality expressed by

$$\frac{q'}{2\sqrt{1+q}}\int e'^2 dv\ , \qquad \textbf{(236)}$$

corresponding to the secular variation of e', the eccentricity of the terrestrial orbit.

728. The first of the equations (234) determines the inclination of the lunar orbit on the plane of the ecliptic. Its integral is

$$g = \left\{H\left(g - \frac{dq}{dv}\right)\right\}^{-\frac{1}{2}}.$$

H being an arbitrary constant quantity. Hence it appears that the inclination is subject to a secular inequality; but as it is quite insensible, the inclination g may be regarded as constant, which is the reason why the most ancient observations do not indicate any change in the inclination of the lunar orbit on the plane of the ecliptic, although the position of the ecliptic has varied sensibly during that interval.

The Mean Longitude of the Moon

729. When the square root is extracted, equation (211) becomes,

$$dt = \frac{dv}{h^2\left(u^2 + 2u\delta u + \delta u^2\right)}\left\{1 - \frac{3m}{h^2}\int\frac{u'^3 dv}{u^4}\sin\left(2v - 2v'\right) + \frac{3m}{2h^4}\left(\int\frac{u^3 dv}{u^4}\sin\left(2v - 2v'\right)\right)^2 - \&\text{c.}\right\};$$

and, making the necessary substitutions there will result [20]

$$
\begin{aligned}
dt = \frac{a^2 dv}{\sqrt{\bar{a}}}\{ & x_0 + x_1 e \cos(cv - v) \\
& + x_2 e^2 \cos(2cv - 2v) \\
& + x_3 e^3 \cos(3cv - 3v) \\
& + x_4 g^2 \cos(2gv - 2q) \\
& + x_5 eg^2 \cos(2gv - cv - 2q + v) \\
& + x_6 eg^2 \cos(2gv + cv - 2q - v) \\
& + x_7 \cos(2v - 2mv) \\
& + x_8 e \cos(2v - 2mv - cv + v) \\
& + x_9 e \cos(2v - 2mv + cv - v) \\
& + x_{10} e' \cos(2v - 2mv + c'mv - v') \\
& + x_{11} e' \cos(2v - 2mv - c'mv + v') \\
& + x_{12} e' \cos(c'mv - v') \\
& + x_{13} ee' \cos(2v - 2mv - cv + c'mv + v - v') \\
& + x_{14} ee' \cos(2v - 2mv - cv - c'mv + v + v') \\
& + x_{15} ee' \cos(cv + c'mv - v - v') \\
& + x_{16} ee' \cos(cv - c'mv - v + v') \\
& + x_{17} e^2 \cos(2cv - 2v + 2mv - 2v) \\
& + x_{18} g^2 \cos(2gv - 2v + 2mv - 2q) \\
& + x_{19} e'^2 \cos(2c'mv - 2v') \\
& + x_{20} \frac{a}{a'} \cos(v - mv) \\
& + x_{21} \frac{a}{a'} e' \cos(v - mv + c'mv - v') \}.
\end{aligned} \qquad \textbf{(237)}
$$

730. The coefficients of which are[21]

$$
\begin{aligned}
x_0 &= 1 + \frac{27m^4}{64(1-m)^2} + \frac{3m^2 \cdot A_0}{4(1-m)} + \tfrac{3}{2}\{A_2^2 + A_1^2 e^2\} \\
x_1 &= -2\left(1 - \tfrac{1}{4} g^2\right) + \frac{15m^2}{4(1-m)} A_1 + 3A_0 \cdot A_1 \\
x_2 &= \tfrac{3}{2} + \tfrac{1}{4} e^2 - \tfrac{3}{2} g^2 - 2A_{10} \\
x_3 &= -1
\end{aligned}
$$

$$x_4 = \tfrac{1}{2}\left(1 + \tfrac{3}{2}e^2 - \tfrac{1}{2}g^2 - 2A_{12} + 3A_{15}e^2\right)$$

$$x_5 = -\tfrac{3}{4} - 2A_{15}$$

$$x_6 = -\tfrac{3}{4}$$

$$x_7 = -\frac{3m^2\left(1 + 2e^2 + \tfrac{5}{2}e'^2\right)}{4(1-m)} - 3m^2e^2\left\{\frac{1+m}{2-2m-c} + \frac{1-m}{2-2m+c}\right\}$$
$$-2A_0\left(1 + \tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3e^2A_1 + 3e^2A_2$$

$$x_8 = +\frac{3m^2\left(1 + 2e^2 - \tfrac{1}{4}g^2 - \tfrac{5}{2}e'^2\right)}{4(1-m)} + \frac{3m^2(1+m)\left(1 + \tfrac{3}{4}e^2 - \tfrac{1}{4}g^2 - \tfrac{5}{2}e'^2\right)}{2-2m-c}$$
$$-\frac{3m^2e^2\left(10 + 19m + 8m^2\right)}{8(2c-2+2m)} - 2A_1\left(1 + \tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3A_0 + 3e^2A_{11}$$

$$x_9 = \frac{3m^2}{4(1-m)} + \frac{3m^2(1-m)}{2-2m+c} - 2A_2 + 3A_0 - 3A_1e^2$$

$$x_{10} = \frac{3m^2}{4(2-m)} - 2A_3 + 3A_6e^2$$

$$x_{11} = -\frac{21m^2}{4(2-3m)} - 2A_4 + 3A_7e^2$$

$$x_{12} = -3m\left\{4A_0 + A_3 - A_4 - 10A_1e^2 + \tfrac{5}{2}(A_7 - A_6)e^2\right\}$$
$$+\left\{\frac{3m^2A_0}{4} + \frac{27m^4}{32(1-m)}\right\}\left\{\frac{7}{2-3m} - \frac{1}{2-m}\right\}$$
$$+\left\{\frac{3m^2}{4(1-m)} + 3A_0\right\}\left\{A_3 + A_4\right\} - 2A_5\left(1 + \tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right)$$
$$+3(A_8 + A_9)e^2 + 3A_1(A_6 + A_7)e^2$$
$$+\frac{3m^2}{4}\left\{11C_6 + 2C_9 - 2C_{10}\right\}$$

$$x_{13} = -\frac{3m^2(2+m)}{4(2-m-c)} - \frac{3m^2}{4(2-m)} - 2A_6 + 3A_3$$

$$x_{14} = \frac{21m^2(2+3m)}{4(2-3m-c)} + \frac{21m^2}{4(2-3m)} - 2A_7 + 3A_4$$

$$x_{15} = -2A_8 + 3A_5$$

$$x_{16} = -2A_9 + 3A_5$$

$$x_{17} = +\frac{3m^2\left(10+19m+8m^2\right)}{8(2c-2+2m)} - \frac{3m^2(1+m)}{2-2m-c} - \frac{9m^2}{16(1-m)}$$
$$-3A_0 + 3A_1 - 2A_{11} - \frac{3m^2A_{10} + \frac{15}{4}m^2A_1^2}{2c-2+2m}$$
$$x_{18} = \frac{3m^2(2+m)}{8(2g-2+2m)} - \frac{3m^2}{16(1-m)} - 2A_{13} - \tfrac{3}{4}A_0 - \frac{3m^2A_{12}}{2g-2+2m}$$
$$x_{19} = -A_{14}$$
$$x_{20} = -\frac{3m^2}{8(1-m)} + \frac{3m^2(5+3m)}{4(1-m)}A_{17} - 2A_{17}\left(1+\tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3A_0 \cdot A_{17}$$
$$x_{21} = -A_{18}.$$

731. Now if quantities of the order m^4 be omitted,

$$\frac{a^2dv}{\sqrt{\bar{a}}}x_0 \text{ becomes } \frac{a^2dv}{\sqrt{\bar{a}}};$$

but in this case equation (230) is reduced to

$$\frac{1}{a} = \frac{1}{\bar{a}}\left\{1 - \frac{m^2}{2} - \frac{3m^2}{4}e'^2\right\},$$

because m^2 differs very little from $\bar{m}^2$, whence

$$\left(\frac{a}{\bar{a}}\right)^2 = 1 + m^2 + \tfrac{3}{2}m^2e',$$

and

$$\frac{a^2dv}{\sqrt{\bar{a}}} = (\bar{a})^{\frac{3}{2}}\left\{\left(1+m^2\right) + \tfrac{3}{2}m^2e'^2\right\}dv, \qquad \textbf{(238)}$$

so that $\frac{a^2dv}{\sqrt{\bar{a}}}$ varies with e', the eccentricity of the terrestrial orbit; but if that variation be omitted, the part that is not periodic of

$$\frac{a^2dv}{\sqrt{\bar{a}}} = (\bar{a})^{\frac{3}{2}}\left(1+m^2\right).dv.$$

If the action of the sun be omitted $a = \bar{a}$, and if $\frac{1}{n}$ be put for $a^{\frac{3}{2}}$, then the part that is not periodic becomes

$$\frac{a^2 dv}{\sqrt{\bar{a}}} = \frac{dv}{n} = a^{\frac{3}{2}}\left(1+m^2\right).dv,$$

and equation (238) is transformed to

$$\frac{a^2 dv}{\sqrt{\bar{a}}} = \frac{dv}{n} + \frac{3m^2}{2n} e'^2 dv,$$

and

$$dt = \frac{a^2 dv}{\sqrt{\bar{a}}} x_0$$

becomes

$$ndt = dv + \tfrac{3}{2} m^2 e'^2 dv,$$

the integral of which is

$$nt + \epsilon = v + \tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) dv,$$

$\bar{e}^2$ being a constant quantity equal to the eccentricity of the earth's orbit at the epoch.

732. Thus the mean longitude of the moon is affected by a secular inequality, occasioned by the variation of the eccentricity of the earth's orbit, and the true longitude of the moon in functions of her mean longitude contains the secular inequality

$$-\tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) dv, \text{ or } -\tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) ndt,$$

called the acceleration; hence the secular inequalities in the mean longitude of the moon, in the longitude of her perigee and nodes, are as the three quantities

$$3\bar{m}^2, \quad -\frac{p'}{\sqrt{1-p}}, \quad \frac{q'}{\sqrt{1+q}}.$$

It is true that the terms depending on the squares of the disturbing force alter the value of the secular equations in the mean longitude a little; but the terms of this order that have a considerable influence on the secular equation of the perigee have but little effect on that of the mean motion.

733. Thus the integral of equation (237) is

$$
\begin{aligned}
nt+\in =+v+\tfrac{3}{2}m^{2}\int\left(e'^{2}-\bar{e}^{2}\right)dv \\
&+C_{0}e\sin\left(cv-v\right) \\
&+C_{1}e^{2}\sin\left(2cv-2v\right) \\
&+C_{2}e^{3}\sin\left(3cv-3v\right) \\
&+C_{3}g^{2}\sin\left(2gv-2q\right) \\
&+C_{4}eg^{2}\sin\left(2gv-cv-2q+v\right) \\
&+C_{5}eg^{2}\sin\left(2gv+cv-2q-v\right) \\
&+C_{6}\sin\left(2v-2mv\right) \\
&+C_{7}e\sin\left(2v-2mv-cv+v\right) \\
&+C_{8}e\sin\left(2v-2mv+cv-v\right) \\
&+C_{9}e'\sin\left(2v-2mv+c'mv-v'\right) \\
&+C_{10}e'\sin\left(2v-2mv-c'mv+v'\right) \\
&+C_{11}e'\sin\left(c'mv-v'\right) \\
&+C_{12}ee'\sin\left(2v-2mv-cv+c'mv+v-v'\right) \\
&+C_{13}ee'\sin\left(2v-2mv-cv-c'mv+v+v'\right) \\
&+C_{14}ee'\sin\left(cv+cmv-v-v'\right) \\
&+C_{15}ee'\sin\left(cv-cmv-v+v'\right) \\
&+C_{16}e^{2}\sin\left(2cv-2v+2mv-2v\right) \\
&+C_{17}g^{2}\sin\left(2gv-2v+2mv-2q\right) \\
&+C_{18}e'^{2}\sin\left(2c'mv-2v'\right) \\
&+C_{29}\frac{a}{a'}\sin\left(v-mv\right) \\
&+C_{20}\frac{a}{a'}e'\sin\left(v-mv+c'mv-v'\right).
\end{aligned}
\qquad \textbf{(239)}
$$

734. If the differential of this equation be compared with equation (237), the following values will be obtained for the indeterminate coefficients–

$$C_{0}=\frac{x_{1}}{c} \qquad C_{10}=\frac{x_{11}}{2-3m}$$

$$C_{1}=\frac{x_{2}}{2c} \qquad C_{11}=\frac{x_{12}}{m}$$

$$C_{2}=\frac{x_{3}}{3c} \qquad C_{12}=\frac{x_{13}}{2-m-c}$$

$$C_3 = \frac{x_4}{2g} \qquad C_{13} = \frac{x_{14}}{2-3m-c}$$

$$C_4 = \frac{x_5}{2g-c} \qquad C_{14} = \frac{x_{15}}{c+m}$$

$$C_5 = \frac{x_6}{2g+c} \qquad C_{15} = \frac{x_{16}}{c-m}$$

$$C_6 = \frac{x_7}{2-2m} \qquad C_{16} = \frac{x_{17}}{2c-2+2m}$$

$$C_7 = \frac{x_8}{2-2m-c} \qquad C_{17} = \frac{x_{18}}{2g-2+2m}$$

$$C_8 = \frac{x_9}{2-2m+c} \qquad C_{18} = \frac{x_{19}}{m}$$

$$C_9 = \frac{x_{10}}{2-m} \qquad C_{19} = \frac{x_{20}}{1-m}$$

$$C_{20} = -2A_{18}\,.$$

Notes

[1] Aristarchus of Samos, c. 310-230 BC, Alexandrian astronomer called the "ancient Copernicus" (see note 1, *Bk. II, Chap. I*), maintained that the Earth moves round the Sun. Only one of his works *On the Sizes and Distances of the Sun and Moon* survives. In it Aristarchus calculates the sun's diameter at about 19 times the moon's diameter, and the sun's distance at 19 times the moon's distance. Although Aristarchus' heliocentric writings did not survive, Archimedes wrote: "*Aristarchus of Samos has published in outline certain hypotheses…He supposes that the fixed stars and the sun are immovable, but that the earth is carried round the sun in a circle…*" (Dreyer, J. E. L., *A History of Astronomy from Thales to Kepler*, 2nd ed., Dover Publications, p. 152, 1953). The lack of stellar parallax counter-argument against the heliocentric model is answered by Aristarchus by assuming the sphere of the fixed stars as infinitely distant with respect to the earth-sun distance.

[2] i.e. 280 BCE

[3] This *reads* "anomalastic" in the 1st edition (published erratum).

[4] See note 3.

[5] *syzigy*. Also spelled "syzygy." The nearly straight-line configuration of three celestial bodies (as the sun, moon, and earth during a solar or lunar eclipse) in a gravitational system. *Merriam-Webster's Collegiate Dictionary*.

[6] This *reads* "385th part" in the 1st edition.

[7] Damoiseau, Théodore, baron de, 1768-1846, *Tables de la lune, formées par la seule théorie de l'attraction, et suivant la division de la circonférence en 360 degrés; par m. le baron de Damoiseau ... Publiées par le Bureau des longitudes*, Paris, Bachelier (successeur de mme. Ve Courcier) 1828.

[8] Left hand side *reads* $\frac{m'a^3}{a'^3} \cdot \frac{a_{/}}{\bar{a}}$ in 1st edition.

[9] The numerator in the second term *reads* $m'(xx+yy+zz)$ in the 1st edition.

[10] Spelled "develope" in the 1st edition.

[11] The 1st edition spelling is, "immoveable".

[12] *apsis*. The point in an astronomical orbit at which the distance of the body from the center of attraction is either greatest or least. *Merriam-Webster's Collegiate Dictionary*.

[13] The right-hand factor $h^3 dv$ *reads* $h^3 dt$ in the 1st edition.

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[14] The argument of the 3[rd] term *reads* $2'cmv - 2v' + 2c'mf$ in the 1[st] edition.

[15] This *reads* "developments" in the 1[st] edition.

[16] Period added after equation (222).

[17] A parenthesis is omitted in the 1[st] term in b_{19} and in the 1[st] edition *reads* as follows:

$$+\{\tfrac{9}{8}(1-2m)\left(1+2e^2+2e'^2\right)+\frac{3(1-2m)\left(1+\tfrac{9}{2}e^2+2e'^2\right)}{4(1-m)}+\frac{3(1+m)}{2(1-m)}\times A_{18}e'^2$$

[18] This is spelled "developement" in the 1[st] edition.

[19] In the 1[st] edition the previous equation is also numbered (234). We retain the original numbering.

[20] The 4[th] term in the argument of x_{14} *reads* $-cmv$ in the 1[st] edition.

[21] The closing parenthesis is missing in the equation for x_4 in the 1[st] edition.