BOOK II

CHAPTER XIV

NUMERICAL VALUES OF THE PERTURBATIONS[1]

619. THE epoch assumed for this computation is that of the French Tables, namely, the 31st of December, at midnight, 1749, mean time at Paris. The data for that epoch are as follow:–

Values of e, e', e'', &c.

Mercury	0.20551320
Venus	0.00688405
The Earth	0.01681395
Mars	0.09305767
Jupiter	0.04807670
Saturn	0.05622460
Uranus	0.04669950

Values of v, v', v'', &c.

Mercury	73°.5661
Venus	127°.9117
The Earth	98°.6211
Mars	331°.473
Jupiter	10°.3511
Saturn	88°.1519
Uranus	166°.614

Values of f, f', f'', &c.

Mercury	7°
Venus	3°. 3931
Mars	1°.8499
Jupiter	1° .3172
Saturn	2°.4986
Uranus	0°.7736

Values of q, q', q'', &c.

Mercury	45°.3452
Venus	74°.4384
Mars	47°.6438
Jupiter	97°.906
Saturn	111°.5064
Uranus	72°.6314

The longitudes are estimated from the mean equinox of spring.

620. The series represented by S and S' in article 453 form the basis of the whole computation, but twelve or fourteen of the first terms of each will be sufficiently correct for all the planets.

The numerical values of the coefficients, A_0, A_1, &c., B_0, B_1, &c., and their differences, for Jupiter and Saturn, are obtained from the formulae in article 455, and those that follow. The mean distances of these two planets are, according to Laplace,

$$a = 5.20116636, \quad a' = 9.5378709,$$

whence

$$\alpha = 0.54531726.$$

$A_0 = 0.228576$	$A_1 = 0.065071$	$A_2 = 0.027012$
$A_3 = 0.012369$	$A_4 = 0.005929$	$A_5 = 0.002918$
$A_6 = 0.001458$	$A_7 = 0.000738$	$A_8 = 0.000376$
$A_9 = 0.000189$	$A_{10} = 0.000091$	$A_{11} = 0.000034$
$\frac{dA_0}{da} = 0.008891$	$\frac{dA_1}{da} = 0.016305$	$\frac{dA_2}{da} = 0.012149$
$\frac{dA_3}{da} = 0.007987$	$\frac{dA_4}{da} = 0.004983$	$\frac{dA_5}{da} = 0.00302$
$\frac{dA_6}{da} = 0.001789$	$\frac{dA_7}{da} = 0.001056$	$\frac{dA_8}{da} = 0.000617$
$\frac{dA_9}{da} = 0.000364$	$\frac{dA_{10}}{da} = 0.000223$	
$\frac{d^2A_0}{da^2} = 0.003314$	$\frac{d^2A_1}{da^2} = 0.002942$	$\frac{d^2A_2}{da^2} = 0.004058$
$\frac{d^2A_3}{da^2} = 0.004070$	$\frac{d^2A_4}{da^2} = 0.003453$	$\frac{d^2A_5}{da^2} = 0.002654$
$\frac{d^2A_6}{da^2} = 0.001919$	$\frac{d^2A_7}{da^2} = 0.001319$	$\frac{d^2A_8}{da^2} = 0.000877$
$\frac{d^2A_9}{da^2} = 0.000559$		

$$\frac{d^3A_0}{da^3} = 0.001466 \qquad \frac{d^3A_1}{da^3} = 0.001556 \qquad \frac{d^3A_2}{da^3} = 0.001551$$

$$\frac{d^3A_3}{da^3} = 0.001868 \qquad \frac{d^3A_4}{da^3} = 0.002061 \qquad \frac{d^3A_5}{da^3} = 0.002013$$

$$\frac{d^3A_6}{da^3} = 0.001808 \qquad \frac{d^3A_7}{da^3} = 0.001478 \qquad \frac{d^3A_8}{da^3} = 0.001156$$

$$\frac{d^4A_0}{da^4} = 0.001069 \qquad \frac{d^4A_1}{da^4} = 0.001064 \qquad \frac{d^4A_2}{da^4} = 0.001107$$

$$\frac{d^4A_3}{da^4} = 0.001138 \qquad \frac{d^4A_4}{da^4} = 0.001284 \qquad \frac{d^4A_5}{da^4} = 0.001808$$

$$\frac{d^4A_6}{da^4} = 0.001503 \qquad \frac{d^4A_7}{da^4} = 0.001469$$

$$\frac{d^5A_0}{da^5} = 0.000993 \qquad \frac{d^5A_1}{da^5} = 0.001001 \qquad \frac{d^5A_2}{da^5} = 0.001011$$

$$\frac{d^5A_3}{da^5} = 0.001044 \qquad \frac{d^5A_4}{da^5} = 0.001088 \qquad \frac{d^5A_5}{da^5} = 0.001175$$

$$\frac{d^5A_6}{da^5} = 0.001212$$

$$B_0 = 0.005026 \qquad B_1 = 0.003674 \qquad B_2 = 0.0024$$

$$B_3 = 0.001493 \qquad B_4 = 0.000904 \qquad B_5 = 0.000537$$

$$B_6 = 0.000315 \qquad B_7 = 0.000183 \qquad B_8 = 0.000107$$

$$B_9 = 0.000062$$

$$\frac{dB_0}{da} = 0.001774 \qquad \frac{dB_1}{da} = 0.000184 \qquad \frac{dB_2}{da} = 0.000162$$

$$\frac{dB_3}{da} = 0.000128 \qquad \frac{dB_4}{da} = 0.000943 \qquad \frac{dB_5}{da} = 0.000661$$

$$\frac{dB_6}{da} = 0.000448 \qquad \frac{dB_7}{da} = 0.000448 \qquad \frac{dB_8}{da} = 0.000293$$

$$\frac{dB_9}{da} = 0.000189$$

$$\frac{d^2B_0}{da^2} = 0.001225 \qquad \frac{d^2B_1}{da^2} = 0.001203 \qquad \frac{d^2B_2}{da^2} = 0.001181$$

$$\frac{d^2B_3}{da^2} = 0.001101 \qquad \frac{d^2B_4}{da^2} = 0.000951 \qquad \frac{d^2B_5}{da^2} = 0.000774$$

$$\frac{d^2B_6}{da^2} = 0.000602 \qquad \frac{d^2B_7}{da^2} = 0.000453$$

$$\frac{d^3B_0}{da^3} = 0.001102 \qquad \frac{d^3B_1}{da^3} = 0.001102 \qquad \frac{d^3B_2}{da^3} = 0.001076$$

$$\frac{d^3B_3}{da^3} = 0.001043 \qquad \frac{d^3B_4}{da^3} = 0.000984 \qquad \frac{d^3B_5}{da^3} = 0.000885$$

$$\frac{d^3B_6}{da^3} = 0.000764$$

Jupiter and Mercury

$$a' = 0.38709812 \qquad a = 5.20116636$$

$$\alpha = 0.0744256$$

$$S = 5.20887 \qquad S' = -0.38683.$$

Jupiter and Venus

$$a' = 0.7233323 \qquad \alpha = 0.13907116$$

$$S = 5.22634 \qquad S' = -0.721579.$$

Jupiter and the Earth

$$a' = 1. \qquad \alpha = 0.19226461$$

$$S = 5.24933 \qquad S' = -0.995358.$$

Jupiter and Mars

$$a' = 1.52369352 \qquad \alpha = 0.29295212$$

$$S = 5.31338 \qquad S' = -1.50717.$$

Jupiter and Uranus

$$a' = 19.183305 \qquad \alpha = 0.2711298$$

$$S = 19.5375 \qquad S' = -5.1528.$$

Secular Variations of Jupiter and Saturn

621. These are given by the numerical values of equations (198), which are computed from the formulae

$$\boxed{4.0} = -\frac{3m'.an\left\{aa'S + \left(a^2 + a'^2\right)S'\right\}}{2\left(a'^2 - a^2\right)^2}$$

$$(4.0) = -\frac{3m'.a^2a'n.S'}{4\left(a'^2 - a^2\right)^2},$$

as the numerical values of all the quantities in these expressions are given, it is easy to find by their substitution, that

$$\begin{aligned} (4.0) &= 0''.000226 \qquad \boxed{4.0} = 0''.000021, \\ (4.1) &= 0''.004291 \qquad \boxed{4.1} = 0''.00744, \\ (4.2) &= 0''.009862 \qquad \boxed{4.2} = 0''.002359, \\ (4.3) &= 0''.22451 \qquad \boxed{4.3} = 0''.001633, \\ (4.5) &= 7''.702 \qquad \boxed{4.5} = 5''.0342, \\ (4.6) &= 0''.09665 \qquad \boxed{4.6} = 0''.03247, \end{aligned} \qquad \textbf{(202)}$$

where the digits 0, 1, 2, 3, &c. refer to Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus.

622. By the substitution of the preceding data, equations (128) and (141), give the following results, when multiplied by the radius reduced to seconds, or, by 206,264″.8, where $\frac{dv}{dt}$ is the sidereal motion of the perihelion of Jupiter in longitude at the epoch 1750, during a period of $365\frac{1}{4}$ days: $2\frac{de}{dt}$ is the annual variation of the equation of the centre: $\frac{df}{dt}$ is the annual inclination[2] of the orbit of Jupiter on the fixed ecliptic of 1750; $\frac{df`}{dt}$ is the annual variation of the inclination on the true ecliptic: $\frac{dq}{dt}$ is the annual and sidereal motion of the ascending node of the orbit of Jupiter on the fixed ecliptic of 1750; and $\frac{dq`}{dt}$ is the same variation with regard to the true ecliptic.

$$\frac{d\bar{v}}{dt} = 6''.5998 \qquad \frac{d\bar{e}}{dt} = 0''.27721 \qquad \frac{d\bar{f}}{dt} = -0''.07814$$

$$\frac{d\bar{f}`}{dt} = -0''.223178 \qquad \frac{d\bar{q}}{dt} = 6''.4562 \qquad \frac{d\bar{q}`}{dt} = -14''.6634.$$

By article 484,

$$(4.0) = \frac{m\sqrt{a}}{m'\sqrt{a'}}(0.4); \qquad \boxed{4.0} = \frac{m\sqrt{a}}{m'\sqrt{a'}}\boxed{0.4};$$

if then, the quantities (202) relating to Jupiter, be multiplied by $\frac{m\sqrt{a}}{m'\sqrt{a'}}$, those corresponding to Saturn will be found, and the formulae (128) give for Saturn

$$\frac{d\bar{v}'}{dt} = 16''.1127 \qquad \frac{d\bar{e}'}{dt} = 0''.54021$$
$$\frac{d\bar{f}'}{dt} = 0''.099741 \qquad \frac{d\bar{q}'}{dt} = -9''.0053.$$

By article 444,

$$f'\sin q' - f\sin q = g \sin\Pi,$$
$$f'\cos q' - f\cos q = g\cos\Pi;$$

and by the substitution of the numerical values of article 613 and 615, it will readily be found, that in 1750

$$\bar{g} = 1^{\circ}\ 15'\ 30'' \qquad \bar{\Pi} = 125^{\circ}\ 44'\ 34'',$$

g being the mutual inclination of the orbits of Jupiter and Saturn, and Π the longitude of the ascending node of the orbit of Saturn on that of Jupiter. If the differential of these equations be taken and the numerical values of

$$\frac{dq'}{dt},\ \frac{dq}{dt},\ \frac{df}{dt},\ \frac{df'}{dt}$$

substituted, it will be found, that

$$\frac{d\bar{g}}{dt} = -0''.000105, \qquad \frac{d\bar{\Pi}}{dt} = -26''.094.$$

623. The variations in the elements that depend on the squares of the disturbing forces must now be computed, and for that purpose the numerical values of P, P', and their differences, must be found from equations (165) and (166).

The coefficients Q_0, Q_1, &c., are given by the expansion of R, article 446; so that[3]

$$Q_0 = -\frac{1}{12}\left\{389A_2 + 201a\cdot\frac{dA_2}{da} + 27a^2\cdot\frac{d^2A_2}{da^2} + a^3\cdot\frac{d^3A_2}{da^3}\right\}$$
$$Q_1 = +\frac{1}{4}\left\{402A_3 + 193a\cdot\frac{dA_3}{da} + 26a^2\cdot\frac{d^2A_3}{da^2} + a^3\cdot\frac{d^3A_3}{da^3}\right\}$$
$$Q_2 = -\frac{1}{4}\left\{396A_4 + 184a\cdot\frac{dA_4}{da} + 25a^2\cdot\frac{d^2A_4}{da^2} + a^3\cdot\frac{d^3A_4}{da^3}\right\}$$
$$Q_3 = +\frac{1}{12}\left\{380A_5 + 174a\cdot\frac{dA_5}{da} + 24a^2\cdot\frac{d^2A_5}{da^2} + a^3\cdot\frac{d^3A_5}{da^3}\right\}$$

$$Q_4 = -\frac{1}{4}aa'\left\{10B_3 + a\frac{dB_3}{da}\right\}$$
$$Q_5 = +\frac{1}{4}aa'\left\{7B_4 + a\frac{dB_4}{da}\right\}.$$

If the values of A_2, A_3, &c. and their differences in article 620, be substituted, then will

$$Q_0 = -2.199192,$$
$$Q_1 = +4.0292,$$
$$Q_2 = -2.43538,$$
$$Q_3 = +0.487332,$$
$$Q_4 = -0.267808,$$
$$Q_5 = +0.139264.$$

With the preceding data, equations (165) and (166) give

$$P = 0.0000114596, \quad P' = -0.000107267;$$

and as the differences of these equations are

$$\frac{dP}{de} = \frac{1}{4}\left\{Q_1 e'^2 \sin(2v' + v) + 2Q_2 ee' \sin(v' + 2v) + 3Q_3 e^2 \sin 3v + Q_5 g^2 \sin(2\Pi + v)\right\},$$
$$\frac{dP'}{de} = \frac{1}{4}\left\{Q_1 e'^2 \sin(2v' + v) + 2Q_2 ee' \cos(v' + 2v) + 3Q_3 e^2 \cos 3v + Q_5 g^2 \cos(2\Pi + v)\right\},$$

&c. &c.

all the quantities in equations (191), (192), (195), and (196), are known; whence for Jupiter at the epoch,[4]

$$(d\bar{v}) = +0''.352941, \qquad (d\bar{e}) = +0''.052278,$$
$$(d\bar{g}) = +0''.000184, \qquad (d\bar{\Pi}) = -0''.007631,$$

and from equations (193) and (194), the corresponding variations in the elements of the orbit of Saturn are

$$(d\bar{v}') = +3''.242722, \qquad (d\bar{e}') = -0''.102763;$$

so that

$$\frac{d\bar{v}}{dt} = 6''.95281; \quad \frac{d\bar{v}'}{dt} = 19''.355448; \quad \frac{d\bar{e}}{dt} = 0''.329487;$$
$$\frac{d\bar{e}'}{dt} = -0''.642968; \quad \frac{d\bar{g}}{dt} = -0''.000079; \quad \frac{d\bar{\Pi}}{dt} = -26''.10163.$$

But at the epoch,

$$\bar{e} = 9916''.53; \quad \bar{e}' = 11597''.1; \quad \bar{v} = 10^\circ.35108;$$
$$\bar{v}' = 88^\circ.15194; \quad \bar{g} = 1^\circ.25838; \quad \bar{\Pi} = 125^\circ.74278.$$

Consequently the elements of the two orbits at any time t are

$$\begin{aligned} e &= 9916''.53 + 0''.329487\,.\,t, \\ v &= 10^\circ.35108 + 6''.95281\,.\,t, \\ e' &= 11597''.1 - 0''.642968\,.\,t, \\ v' &= 88^\circ.15194 + 19''.3555448\,.\,t, \\ g &= 1^\circ.25838 + 0''.000079\,.\,t, \\ \Pi &= 125^\circ.74278 - 26''.10163\,.\,t. \end{aligned} \qquad \textbf{(203)}$$

If $t = 0$ these expressions will give the elements in 1950, and if the computation be repeated with them it will be found that in 1950

$$\frac{dv}{dt} = 7''.053178; \quad \frac{dv'}{dt} = 19''.424739; \quad \frac{de}{dt} = 0''.326172;$$
$$\frac{de'}{dt} = -0''.648499; \quad \frac{dg}{dt} = -0''.001487; \quad \frac{d\Pi}{dt} = -26''.402056.$$

The differences between these and their values for 1750, divided by 200, will be their second differences, therefore the formulae (198), with regard to Jupiter and Saturn, are

$$\begin{aligned} e &= 9916''.53 + 0''.329487\,.\,t - 0''0000082871\,.\,t^2, \\ v &= 10^\circ 21'4'' + 6''.952808\,.\,t + 0''.0002509259\,.\,t^2, \\ e' &= 11597''.1 - 0''.642968\,.\,t - 0''.0000138275\,.\,t^2, \\ v' &= 88^\circ 9'6'' + 19''.3555440\,.\,t + 0''.0001732274\,.\,t^2, \\ g &= 1^\circ 15'30''.2 + 0''.000078\,.\,t - 0''.0000391311\,.\,t^2, \\ \Pi &= 125^\circ 44'33'' - 26''.1028\,.\,t - 0''.0007507307\,.\,t^2, \end{aligned} \qquad \textbf{(204)}$$

which will give the elements of the orbits of these two planets for 1000 or 1200 years before or after 1750.

Periodic Inequalities of Jupiter

624. The inequalities in the radius vector and longitude, which are independent of the eccentricities and inclinations, are computed from

$$\frac{dr}{a} = -\frac{m'}{6}a^2\cdot\frac{dA_0}{da} + \frac{m'}{2}\cdot\Sigma.C_i\ .\cos i\left(n't - nt + \in' - \in\right),$$
$$dv = \frac{m'}{2}\cdot\Sigma.F_i\ .\sin i\left(n't - nt + \in' - \in\right);$$

If $i = 0$, then by articles 536 and 537

$$C_1 = \frac{n^2}{n'\left(2n - n'\right)}\left\{\frac{2n}{n - n'}\cdot aA_1 + a^2\cdot\frac{dA_1}{da}\right\}$$
$$F_1 = \frac{n}{n - n'}\left\{-\frac{2n}{n - n'}\cdot aA_1 + 2C_1\right\}.$$

But

$$n = 109256'';\quad n' = 43996''.7;\quad a = 5.20116626;$$
$$m' = \frac{1}{3359.4};\quad A_1 = 0.0078973;\quad \frac{dA_1}{da} = 0.00531108;$$
$$\frac{2n}{n - n'}\cdot aA_1 = 0.1375352;\quad a^2\cdot\frac{dA_1}{da} = 0.143676;$$
$$\frac{2n}{n - n'}aA_1 + a^2\frac{dA_1}{da} = 0.281209;$$
$$\log 0.281209 = 9.4490293$$
$$\log\frac{2n^2}{n'\left(2n - n'\right)} = 0.4926697$$
$$\log 2C_1 = \overline{9.9416990} = \log 0.874378$$

hence

$$-\frac{n}{n - n'}aA_1 + 2C_1 = 0.8056104.$$

$$\log 0.8056104 = 9.9061248$$
$$\log\frac{n}{n - n'} = 0.2238068$$
$$\text{log of radius in seconds} = \underline{5.3144256}$$
$$\text{the sum is}\ldots\ldots\ldots 5.4443572$$
$$\log 3359.4 = 3.5262617$$
$$\log 82''.812 = 1.9180955$$

Consequently, when $i = 1$, $dv = 82''.821\sin\left(n't - nt + \in' - \in\right)$. Hence if i be made successively equal to all the positive numbers from 1 to 9, and the corresponding quantities substituted in the preceding formulae, it will be found that the inequalities of this order in the longitude and radius vector of Jupiter arising from the action of Saturn, are

$$d\,v = \begin{Bmatrix} +82''.811711\sin\left(n't-nt+\epsilon'-\epsilon\right) \\ -204''.406384\sin 2\left(n't-nt+\epsilon'-\epsilon\right) \\ -17''.071564\sin 3\left(n't-nt+\epsilon'-\epsilon\right) \\ -3''.926319\sin 4\left(n't-nt+\epsilon'-\epsilon\right) \\ -1''.210573\sin 5\left(n't-nt+\epsilon'-\epsilon\right) \\ -0''.42843\sin 6\left(n't-nt+\epsilon'-\epsilon\right) \\ -0''.170923\sin 7\left(n't-nt+\epsilon'-\epsilon\right) \\ -0''.076086\sin 8\left(n't-nt+\epsilon'-\epsilon\right) \\ -0''.041273\sin 9\left(n't-nt+\epsilon'-\epsilon\right) \end{Bmatrix}$$

$$d\,r = \begin{Bmatrix} -0.0000620586 \\ +0.000676876\cos\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.00289662\cos 2\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.0003021367\cos 3\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.0000782514\cos 4\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.0000258952\cos 5\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.0000094779\cos 6\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.000003756\cos 7\left(n't-nt+\epsilon'-\epsilon\right) \\ -0.0000014781\cos 8\left(n't-nt+\epsilon'-\epsilon\right) \\ 0.0000004799\cos 9\left(n't-nt+\epsilon'-\epsilon\right) \end{Bmatrix}.$$

625. The inequalities depending on the first powers of the eccentricities are obtained from

$$\begin{aligned} d\,r &= m'fe\cos\left(nt+\epsilon-v\right)+m'f'e'.\cos\left(nt+\epsilon-v'\right) \\ &\quad + m'e.\sum.D_i.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-v\right\} \\ &\quad + m'e'.\sum.E_i.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-v'\right\}, \\ d\,v &= m'e.\sum.G_i.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-v\right\} \\ &\quad + m'e'.\sum.H_i.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-v'\right\}. \end{aligned}$$

by making successively equal to the whole positive numbers, from 1 to 7, and to the whole negative numbers, from -1 to -5, and substituting the numerical data corresponding to each in the coefficients D_i, E_i, &c., which are given in articles 536 and 537. The values of e and e' at the epoch are sufficiently exact for all the terms of this order, except those having the arguments $2n't - nt + 2\epsilon' - \epsilon$, and $3n' - 2nt + 3\epsilon' - 2\epsilon$, whose periods are so long, that

$$9916''.53 + 0''.329487\,.\,t, \text{ and } 11597''.1 - 0''.642968\,.\,t$$

must be employed instead of e and e'. It will then be found that the perturbations of Jupiter are

$$d\,r = \left\{\begin{array}{l}
+0.0000206111\cos\left(nt + \epsilon - v\right) \\
-0.0000795246\cos\left(n't + \epsilon' - v\right) \\
+0.0000492096\cos\left(n't + \epsilon' - v'\right) \\
-0.000292213\cos\left\{2n't - nt + 2\epsilon' - \epsilon - v\right\} \\
+0.0001688085\cos\left\{2n't - nt + 2\epsilon' - \epsilon - v'\right\} \\
-0.0004584483\cos\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v\right\} \\
+0.0009047822\cos\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v'\right\} \\
+0.0001259429\cos\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v\right\} \\
-0.0002424413\cos\left\{4n't - 3nt + 4\epsilon' - 2\epsilon - v'\right\} \\
+0.0000268383\cos\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v\right\} \\
-0.0000516048\cos\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v'\right\} \\
+0.0000579151\cos\left\{2nt - n't + 2\epsilon - \epsilon' - v\right\} \\
-0.000134653\cos\left\{3nt - 2n't + 3\epsilon - 2\epsilon' - v\right\}
\end{array}\right\}.$$

[and] [5]

$$d\,v = \left\{\begin{array}{l}
+8''.608489\sin\left(n't + \epsilon' - v\right) \\
-9''.692386\sin\left(n't + \epsilon' - v'\right) \\
-\left\{138''.373337 + t\,.\,0''.0045985\right\}\sin\left\{2n't - nt + 2\epsilon' - \epsilon - v\right\} \\
+\left\{56''.634099 - t\,.\,0''.0031398\right\}\sin\left\{2n't - nt + 2\epsilon' - \epsilon - v'\right\} \\
-\left\{44''.460822 + t\,.\,0''.0014775\right\}\sin\left\{3n't - 2nt + 2\epsilon' - 2\epsilon - v\right\} \\
+\left\{84''.942569 - t\,.\,0''.004794\right\}\sin\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v'\right\} \\
+7''.925312\sin\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v\right\} \\
-15''.629621\sin\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v'\right\} \\
+1''.047717\sin\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v\right\} \\
-2''.781664\sin\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v'\right\} \\
+0''.407251\sin\left\{6n't - 5nt + 6\epsilon' - 5\epsilon - v\right\} \\
\quad \text{Continued on next page}
\end{array}\right\}.$$

$$\left\{\begin{array}{l}\text{Continued from previous page}\\ -0''.913302\sin\{6n't-5nt+6\epsilon'-5\epsilon-\varpi'\}\\ +0''.149277\sin\{7n't-6nt+7\epsilon'-6\epsilon-\varpi\}\\ -0''.325592\sin\{7n't-6nt+7\epsilon'-6\epsilon-\varpi'\}\\ -5''.208122\sin\{2n't-nt+2\epsilon-\epsilon'-\varpi\}\\ -0''.569738\sin\{2n't-nt+2\epsilon-\epsilon'-\varpi'\}\\ +12''.87665\sin\{3n't-2nt+3\epsilon-2\epsilon'-\varpi\}\\ -0''.352399\sin\{3n't-2nt+3\epsilon-2\epsilon'-\varpi'\}\\ +1''.287482\sin\{4n't-3nt+4\epsilon-3\epsilon'-\varpi\}\\ -0''.172892\sin\{4n't-3nt+4\epsilon-3\epsilon'-\varpi'\}\\ +0''.356627\sin\{5n't-4nt+5\epsilon-4\epsilon'-\varpi\}\\ -0''.083189\sin\{5n't-4nt+5\epsilon-4\epsilon'-\varpi'\}\end{array}\right\}.$$

Inequalities depending on the Squares of the Eccentricities and Inclinations

626. These are computed by making i successively equal to 1, 2, 3, &c., in formulae (163) and (164).

If $i=1$, that part of the perturbations in longitude, depending on the argument $n't+nt+\epsilon'+\epsilon$, is[6]

$$dv=\frac{1}{\sqrt{1-e^2}}\left\{\begin{array}{l}\dfrac{2d.(rdr)}{a^2.ndt}-\dfrac{m'}{2}\left\{\begin{array}{l}\left(\frac{1}{2}C_1+D_1\right)e^2.\sin\left(n't+nt+\epsilon'+\epsilon-2\varpi\right)\\ +E_1ee'.\sin\left(n't+nt+\epsilon'+\epsilon-\varpi-\varpi'\right)\end{array}\right\}\\ \qquad-\dfrac{m'}{2}\left\{\dfrac{3n^2}{(n'+n)^2}\cdot\Sigma.aN+\Sigma a^2\cdot\dfrac{dN}{da}\cdot\dfrac{2n}{n'+n}\right\}\sin\left(n't+nt+\epsilon'+\epsilon+L\right)\end{array}\right\};$$

where[7]

$$\frac{2d(rdr)}{a^2.ndt}=-\frac{m'.n^2}{n'^2+2nn'}\left\{\begin{array}{l}3\left(\frac{1}{2}C_1+D_1\right)e^2.\sin\left(n't+nt+\epsilon'+\epsilon-2\varpi\right)\\ +3E_1.ee'.\sin\left(n't+nt+\epsilon'+\epsilon-\varpi-\varpi'\right)\\ +\left\{\dfrac{2n}{n'+n}\cdot\Sigma.dN+\Sigma.a^2\dfrac{dN}{da}\right\}\cdot\sin\left(n't+nt+\epsilon'+\epsilon+L\right)\end{array}\right\}.$$

$$C_1=\frac{n^2}{n^2-(n'-n)^2}\cdot\left\{\frac{2n}{n-n'}\cdot aA_1+a^2\frac{dA_1}{da}\right\}$$

$$D_1=\frac{n^2}{n'^2-n^2}\left\{\frac{3n}{n'-n}\cdot aA_1-\frac{(n'-n)(n'-2n)-3n^2}{n^2}\cdot C_1+\tfrac{1}{2}a^3\frac{d^2A_1}{da^2}\right\}$$

$$E_1 = -\frac{n^2}{n'^2 - n^2}\cdot\left\{a^2\frac{dA_0}{da} + \tfrac{1}{2}a^3\frac{d^2A_0}{da^2}\right\}.$$

$$\begin{aligned}&\sum .N\,.\sin\left(n't + nt + \in' + \in - L\right) = \\ &+N_0\,.e^2\,.\sin\left(n't + nt + \in' + \in - 2\varpi\right) \\ &+N_1\,.ee'\,.\sin\left(n't + nt + \in' + \in - \varpi - \varpi'\right) \\ &+N_2\,.e'^2\,.\sin\left(n't + nt + \in' + \in - 2\varpi'\right) \\ &+N_6\,.g^2\,.\sin\left(n't + nt + \in' + \in - 2\Pi\right).\end{aligned}$$

The coefficients N_0, N_1, &c., are given in article 459, and if the numerical values of A_0, A_1, their differences, and also[8] $n = 109256''$, $n' = 43996''.6$, be substituted, it will be found that $d\,v$ takes the form

$$\begin{aligned}d\,v = b\,.\,e^2\,.\sin\left(n't + nt + \in' + \in - 2\varpi\right) \\ +b_1\,.\,ee'\,.\sin\left(n't + nt + \in' + \in - \varpi - \varpi'\right) \\ +b_2\,.\,e'^2\,.\sin\left(n't + nt + \in' + \in - 2\varpi'\right) \\ +b_3\,.\,g^2\,.\sin\left(n't + nt + \in' + \in - 2\Pi\right),\end{aligned}$$

where b, b_1, b_2 and b_3 are given numbers. But $d\,v$ may be expressed by

$$d\,v = P\,.\sin\left(n't + nt + \in' + \in\right) - P'\,.\cos\left(n't + nt + \in' + \in\right),$$

where,[9]

$$P' = be^2\,.\sin 2\varpi + b_1ee'\,.\sin\left(\varpi + \varpi'\right) + b_2\,.\,e'^2\sin 2\varpi' + b_3\,.\,g^2\sin 2\Pi$$

[and]

$$P = be^2\,.\sin 2\varpi + b_1ee'\,.\cos\left(\varpi + \varpi'\right) + b_2\,.\,e'^2\cos 2\varpi' + b_3\,.\,g^2\cos 2\Pi;$$

substituting the values of the elements given in article 619, it will be found by the method in article 569, that

$$\sqrt{P^2 + P'^2} = 1''.004; \qquad \frac{P'}{P} = -\tan 45^\circ.4894 = -\frac{\sin 45^\circ.4894}{\cos 45^\circ.4894}.$$

Consequently the inequality depending on $i = 0$, becomes

$$d\,v = 1''.004\,.\sin\left(n't + nt + \in' + \in + 45^\circ.4894\right).$$

627. It will be found by this method of computation that all the sensible inequalities in longitude and in the radius vector depending on the squares and products of the eccentricities and inclinations, are included in the following expressions; observing that the inequality having the argument $3n't - 5nt + 3\epsilon' - 5\epsilon$, must be computed with the formulae (204), on account of the great length of its period.[10 11]

$$dv = \left\{ \begin{array}{l} +1''.004\sin\left(n't + nt + \epsilon' + \epsilon + 45^\circ.4894\right) \\ -5''.57871\sin\left(2n't + 2\epsilon' + \epsilon + 15^\circ.93999\right) \\ +11''.72425\sin\left(3n't - nt + 3\epsilon' - \epsilon + 79^\circ.6633\right) \\ -18''.07528\sin\left(4n't - 2nt + 4\epsilon' - 2\epsilon - 57^\circ.2072\right) \\ +\left\{169''.2659 - t\,.0''.004277\right\}\sin\left(3n't - 5nt + 3\epsilon' - 5\epsilon + 55^\circ.6802 + t\,.50''.5084\right) \\ +1''.64714\sin\left(6n't - 4nt + 6\epsilon' - 4\epsilon - 54^\circ.43\right) \\ +2''.4764\sin\left(n't - nt + \epsilon - \epsilon' + 43^\circ.2836\right) \\ -5''.288\sin\left(2n't - 2nt + 2\epsilon' - 2\epsilon + 42^\circ.6789\right) \\ +0.000082242\cos\left(2n't + 2\epsilon + 11^\circ.0153\right) \\ +0.000022625\cos\left(3n't - nt + 3\epsilon' - 2\epsilon - 21^\circ.7884\right) \\ -0.0001010533\cos\left(4n't - 2nt + 4\epsilon' - 2\epsilon - 51^\circ.0677\right) \\ -\left\{0.00211145 - t\,.\,0.00000005323\right\}\cos\left(3n't - 5nt + 3\epsilon' - 5\epsilon + 55^\circ.597 + 50''.4144\,.\,t\right) \\ -0.0000652204\cos\left(2n't - 2nt + 2\epsilon' - 3\epsilon + 54^\circ.1477\right) \end{array} \right\}.$$

Perturbations depending on the Third Powers and Products of the Eccentricities and Inclinations

628. These are contained in equation (172). But, in order to find the numerical value of the principal term, the differences of P and P' must be computed. By article 623,

$$P = 0.0000114596, \qquad P' = -0.000107267$$

are the values of these quantities in 1750; but their values in the years 2250, and 2750, will be obtained by making t successively equal to 500 and 1000, in equations (204); whence the elements of the orbits of Jupiter and Saturn at these two periods will be known; and if the same computation that was employed for the determination of P and P' be repeated with them, the results in 2250, and 2750, will be

$$\begin{array}{l} P = -0.000008407 \text{ [in 2250]} \\ P' = -0.00010552 \text{ [in 2250]} \\ P = -0.000027365 \text{ [in 2750]} \\ P' = -0.00010009 \text{ [in 2750]}; \end{array}$$

and, by the method of article 480

$$\frac{dP}{dt} = -0.000000040645 \text{ [in 2250]};$$
$$\frac{dP'}{dt} = -0.0000000002249 \text{ [in 2250]};$$
$$\frac{dP}{dt} = -0.000000000003642 \text{ [in 2750]};$$
$$\frac{dP'}{dt} = -0.000000000014865 \text{ [in 2750]};$$

with these data the principal term of the great inequality put under the form of equation (171) becomes

$$d\,v = \left\{ \begin{array}{l} +1263''.79967 - 0''.008418\,.\,t - 0''.00001925\,.\,t^2 \sin\left(5n't - 2\,nt + 5\in' - 2\in\right) \\ +119''.52695 - 0''.473686\,.\,t - 0''.000078562\,.\,t^2 \cos\left(5n't - 2\,nt + 5\in' - 2\in\right) \end{array} \right\}.$$

In order to compute the inequality[12]

$$d\,v = -\frac{2m'n}{5n' - 2n}\left\{ a^2 \cdot \frac{dP}{da} \cdot \cos\left(5n't - 2\,nt + 5\in' - 2\in\right) - a^2 \cdot \frac{dP'}{da} \cdot \sin\left(5n't - 2\,nt + 5\in' - 2\in\right) \right\}$$

equation (165), gives[13]

$$\begin{aligned} 4\frac{dP}{da} &= +\frac{dQ_0}{da} \cdot e^3 \sin 3v' + \frac{dQ_1}{da} \cdot e'^2 \sin\left(v' + v\right) \\ &\quad + \frac{dQ_2}{da} \cdot e'e^2 \sin\left(v + 2v'\right) + \frac{dQ_3}{da} \cdot e^3 \sin 3v \\ &\quad + \frac{dQ_4}{da} \cdot e'g^2 \sin\left(2\Pi + v'\right) + \frac{dQ_5}{da} \cdot eg \sin\left(2\Pi + v\right). \end{aligned}$$

The quantities $\frac{dQ_0}{da}$, &c. are obtained from the values of Q_0, Q_1, &c. in article 623.[14]

With which and the numerical values of the elements at the epoch 1750, the preceding value of $\frac{dP}{da}$ gives

$$-\frac{2m'.n}{5n' - 2n} a^2 \cdot \frac{dP}{da} = -17''.22886;$$

and, by changing the sines into cosines, the same expression gives

$$\frac{2m'n}{5n'-2n}a^2\cdot\frac{dP'}{da}=5''.360016\,.$$

If t be made equal to 200 in the equations (204), and the computation repeated with the resulting values of the elements, it will be found that in 1950

$$-\frac{2m'.n}{5n'-2n}a^2\cdot\frac{dP}{da}=-16''.836801$$
$$\frac{2m'n}{5n'-2n}a^2\cdot\frac{dP'}{da}=6''.449839;$$

but

$$\frac{-17''.22886+16''.83680}{200}=-0''.0019603,$$

and[15]

$$\frac{6''.449839-5''.360016}{200}=0''.0004491;$$

hence[16]

$$d\,v=-\{17''.228862-0''.0019603\,.\,t\}\,.\sin\left(5n't-2\,nt+5\in'-2\in\right)$$
$$+\{5''.360016+0''.0004491\,.\,t\}\,.\cos\left(5n't-2\,nt+5\in'-2\in\right).$$

The only remaining inequalities of this order are,[17]

$$-m'Ke\,.\sin\left(5n't-2\,nt+5\in'-2\in-\nu+B\right)$$
$$+\frac{5m'}{4}\cdot K'e\,.\sin\left(5n't-4\,nt+5\in'-4\in+\nu+B\right)$$

the numerical values of which may easily be found equal to

$$d\,v=\left(0''.8203-0''.00059324\,.\,t\right).\sin\left(5n't-2\,nt+5\in'-2\in\right)$$
$$-\left(1''.83796-0''.00000149\,.\,t\right).\cos\left(5n't-2\,nt+5\in'-2\in\right)$$
$$+10''.0847\,.\sin\left(4nt-5n't+4\in'-5\in-45^\circ.36225\right).$$

The great inequality of Jupiter also contains the terms

$$d\,v=\left(12''.5365-0''.001755t\right).\sin\left(5n't-2\,nt+5\in'-2\in\right)$$
$$-\left(8''.1211+0''.004885t\right).\cos\left(5n't-2\,nt+5\in'-2\in\right);$$

depending on the fifth powers and products of the eccentricities and inclinations, the computation of these is exactly the same with the examples given, but very tedious on account of the form of the coefficients of the series *R*. If all the terms depending on the argument $5n't - 2nt + 5\epsilon' - 2\epsilon$ be collected, it will be found that the great inequality of Jupiter is

$$dv = \left\{ \begin{array}{l} +\left\{1261''.56 - 0''.013495 \,.\, t - 0''.00001925 \,.\, t^2\right\} . \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) \\ +\left\{96''.4661 - 0''.47466 \,.\, t - 0''.00007856 \,.\, t^2\right\} . \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) \end{array} \right\}.$$

Inequalities depending on the Squares of the Disturbing Force

629. These are given by equations (182) and (199): their numerical values are

$$dv = +4''.0248 \,.\, \sin\left(5nt - 10n't + 5\epsilon - 10\epsilon' + 51^\circ.3653\right)$$
$$-13''.2389 \sin\left(\text{twice the argument of the great inequality of Jupiter}\right).$$

The inequality mentioned in article 589, according to Pontécoulant,[18] is [for Jupiter],

$$dz = 2''.16304 \,.\, \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) + 16''.9712 \times \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right);$$

and [for Saturn]

$$dz' = 3''.4645 \,.\, \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) - 40''.3437 \times \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right).$$

Periodic Inequalities in the Radius Vector, depending on the Third Powers and Products of the Eccentricities and Inclinations

630. These are occasioned by Saturn, and are easily found from equation (168) to be [19]

$$dr = \left\{ \begin{array}{l} -0.0003042733 \,.\, \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon - 12^\circ.14694\right) \\ +0.0001001860 \,.\, \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon + 45^\circ.27972\right) \end{array} \right\}.$$

Periodic Inequalities in Latitude

631. These are obtained from equations (160) and (177).

$$f = 1^\circ.3172,$$

is the inclination of Jupiter's orbit on the fixed ecliptic of 1750,

$$\frac{df}{dt} = -0''.07821 \text{ is its secular variation,}$$

and

$$\frac{df`}{dt} = -0''.22325,$$

is the same, with regard to the variable ecliptic; also

$$q = 97^\circ.906,$$

is the longitude of the ascending node of Jupiter's orbit on the fixed ecliptic; $\frac{dq}{dt} = 6''.4571,$ is its secular variation with regard to that plane, and $\frac{df`}{dt} = -14''.6626$ is its secular variation with regard to the variable ecliptic. Equations (197) give

$$(df) = -0.0000726, \text{ and } (dq) = 0.0008113,$$

for the variations depending on the squares of the disturbing forces; hence

$$\frac{df}{dt} = -0''.078283, \ \frac{dq}{dt} = 6''.457,$$

with regard to the fixed ecliptic, and

$$\frac{df`}{dt} = -0''.22325, \ \frac{dq`}{dt} = -14''.6626.$$

With these it will be found that

$$ds = \left\{ \begin{array}{l} +0''.564458 . \sin(n't + \epsilon' - \Pi) \\ +0''.663927 . \sin(2n't - nt + 2\epsilon' - \epsilon - \Pi) \\ +1''.119782 . \sin(3n't - 2nt + 3\epsilon' - 2\epsilon - \Pi) \\ -0''.279382 . \sin(4n't - 3nt + 4\epsilon' - 3\epsilon - \Pi) \\ -0''.26913 . \sin(2nt - n't + 2\epsilon - \epsilon' - \Pi) \\ +3''.94168 . \sin(3nt - 5n't + 3\epsilon - 5\epsilon' + 59^\circ.50097) \end{array} \right\};$$

which are the only sensible inequalities in the latitude of Jupiter.

632. The action of the earth occasions the inequalities

$$dv = \begin{Bmatrix} +0''.120833 . \sin(n't - nt + \epsilon' - \epsilon) \\ -0''.000086 . \sin 2(n't - nt + \epsilon' - \epsilon) \end{Bmatrix}$$

in the longitude of Jupiter, n' being the mean motion of the earth, and the action of Uranus is the cause of the following perturbations in the longitude of Jupiter,[20]

$$dv = \begin{Bmatrix} +1''.051737 . \sin 2(n't - nt + \epsilon' - \epsilon) \\ -0''.427296 . \sin 2(n't - nt + \epsilon' - \epsilon) \\ -0''.044085 . \sin 3(n't - nt + \epsilon' - \epsilon) \\ -0''.005977 . \sin 4(n't - nt + \epsilon' - \epsilon) \\ +0''.123506 . \sin(nt + \epsilon - v) \\ -0''.23524 \ . \sin(nt + \epsilon - v') \\ -0''.53308 \ . \sin(2n't - nt + 2\epsilon' - \epsilon - v) \\ +0''.102673 . \sin(2n't - nt + 2\epsilon' - \epsilon - v') \\ -0''.127963 . \sin(3n't - 2nt + 3\epsilon' - \epsilon - v') \end{Bmatrix}$$

where n' is the mean motion of Uranus.

These are all the inequalities that are sensible in the motions of Jupiter; those of Saturn may be computed in the same manner.

On the Laws, Periods, and Limits of the Variations in the Orbits of Jupiter and Saturn

633. When the values of p, p', q, q', are substituted in equations (137) they give

$$gN = (4.5)(N' - N); \quad gN' = (5.4)(N - N');$$

and as

$$(5.4) = (4.5)\frac{m\sqrt{a}}{m'\sqrt{a'}},$$

$$g^2 + g\left\{\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}\right\}(4.5) = 0 .$$

The roots of which are,

$$g_1 = 0; \quad g = -\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}(4.5)$$

so that equations (138) become

$$\begin{aligned} p &= N.\sin(gt+x) + N_/ .\sin x_/ \\ q &= N.\cos(gt+x) + N_/ .\cos x_/ \\ p' &= N'.\sin(gt+x) + N_/ .\sin x_/ \\ q' &= N'.\cos(gt+x) + N_/ .\cos x_/ . \end{aligned} \qquad \textbf{(205)}$$

Whence,

$$p' - p = (N' - N)\sin(gt+x); \quad q' - q = (N' - N)\cos(gt+x),$$

and at the epoch when $t = 0$

$$\tan x = \frac{p' - p}{q' - q}.$$

But as

$$N' = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot N;$$

and

$$p' - p = (N' - N)\sin x,$$

so

$$N = -\frac{m'\sqrt{a'}(p' - p)}{\left(m\sqrt{a} + m'\sqrt{a'}\right)\sin x}.$$

Again, by article 504

$$m\sqrt{a}.\ p + m'\sqrt{a'}.\ p' = \text{constant},$$
$$m\sqrt{a}.\ q + m'\sqrt{a'}.\ q' = \text{constant};$$

or in consequence of

$$Nm\sqrt{a} + N'm'\sqrt{a'} = 0$$

$$\left(m\sqrt{a} + m'\sqrt{a'}\right)N_/ .\sin x_/ = \text{constant},$$
$$\left(m\sqrt{a} + m'\sqrt{a'}\right)N_/ .\cos x_/ = \text{constant}.$$

Whence[21]

$$\tan x_/ = \frac{m\sqrt{a}.\ p + m'\sqrt{a'}.\ p'}{m\sqrt{a}.\ q + m'\sqrt{a'}.q'},$$

and

$$N_/ = \frac{m\sqrt{a}.\ p + m'\sqrt{a'}.\ p'}{\left(m\sqrt{a} + m'\sqrt{a'}\right)\sin x_/}$$

and as at the epoch

$$p = \tan \bar{f}.\sin\bar{q} \qquad q = \tan\bar{f}.\cos\bar{q}$$
$$p' = \tan\bar{f}'.\sin\bar{q}' \qquad q' = \tan\bar{f}'.\cos\bar{q}'$$

are given, all the constant quantities g, $g_{/}$, x, $x_{/}$, N, N', and $N_{/}$, are obtained from the preceding equations.

The variations in the inclinations are at their maxima and minima when $gt+x-x_{/}$ is either zero or 180°; hence if $x_{/}$ be substituted for $gt+x$, equations (205) give

$$\tan f = N+N_{/}; \qquad \tan f' = N'+N_{/}$$

for the maxima of the inclinations; and when $x_{/}+180^{\circ}$ is put for $gt+x$, they give for the minima,[22]

$$\tan f = N-N_{/}; \qquad \tan f' = N'-N_{/}.$$

The maxima and minima of the longitude of the nodes are given by the equations $dq=0$, $dq'=0$, or $d\,.\tan q=0$, whence

$$q\frac{dp}{dt}-p\frac{dq}{dt}=0,$$

and therefore $pp'+qq'=p^2+q^2$, and by the substitution of the quantities in equations (205), it becomes

$$N+N_{/}\,.\cos\left(gt+x-z_{/}\right)=0,$$

or

$$\cos\left(gt+x-x_{/}\right)=-\frac{N}{N_{/}}.$$

If $N_{/}$ be greater than N independently of the signs, the nodes will have a libratory[23] motion; but if $N_{/}$ be less than N, they will circulate in one direction.

$\text{Tan}\, f=\sqrt{N_{/}^2-N^2}$ corresponds to the preceding value of

$$\cos\left(gt+x-x_{/}\right);$$

it gives the inclination corresponding to the stationary points of the node.

These points are attained when

$$\cos\left(gt+x-x_{/}\right)=-\frac{N}{N_{/}},$$

whereas the maxima and minima of the inclinations happen when

$$\cos\left(gt + x - x_{\prime}\right) = \pm 1 .$$

The stationary positions of the nodes therefore do not correspond either to the maxima or minima of the inclination, or to the semi-intervals between them.

In 1700, by Halley's Tables,[24] [25]

$$f = 1^{\circ}\,19'\,10'' \qquad q = 97^{\circ}\,34'\,9''$$
$$f' = 2^{\circ}\,30'\,10'' \qquad q' = 101^{\circ}\,5'\,6''$$

hence at that time,

$$p = 0.02283 \qquad q = -\,0.00303$$
$$p' = 0.04078 \qquad q' = -\,0.01573,$$

with these values, Mr. Herschel[26] found

$$N_{\prime} = 0.02905 \qquad N' = 0.01537 \qquad N = -0.00661$$
$$x = 125^{\circ}\,15'\,40'' \quad x_{\prime} = 103^{\circ}\,38'\,40'' \quad g = -25''.5756,$$

consequently for Jupiter

$$\tan f = 0.029880\sqrt{1 - 0.43290\cos\left\{21^{\circ}\;37' - t \times 25''.5756\right\}}$$

and for Saturn

$$\tan f' = 0.03287\sqrt{1 + 0.82665\cos\left\{21^{\circ}\;37' - t \times 25''.5756\right\}} .$$

Also

$$N_{\prime} + N' = 0.04442 \qquad N_{\prime} - N' = 0.01368;$$

so that the maxima and minima of the inclinations of Saturn's orbit are $2^{\circ}\,32'\,40''$ and $0^{\circ}\,47'$, and its greatest deviation from its mean state does not exceed $52^{\circ}\,50''$. In Jupiter's orbit, the maximum is $2^{\circ}\,2'\,30''$, and the minimum $1^{\circ}\,17'\,10''$, and the greatest deviation from a mean state is $0^{\circ}\,22'\,40''$.

The longitude of the node q has a maximum and minimum in both orbits, because $N_{\prime} \rangle N'$. The extent of its librations in Jupiter's orbit will be $13^{\circ}\,9'\,40''$, and in Saturn's $31^{\circ}\,56'\,20''$, on either side of its mean station on the plane of the ecliptic supposed immovable.[27] The period in which the inclinations vary from their greatest to their least values, and the nodes from their greatest to their least longitudes, is by article 486

$$= \frac{360^{\circ}}{g} = \frac{360^{\circ}}{25''.5756} = 50{,}673 \text{ Julian years} .$$

634. The limits and periods of the variations in the eccentricities and longitudes of the perihelia are obtained by a similar process, from equations (133), and those in article 485. The quantities

$$h = e\sin v\ , \quad l = e\cos v\ , \quad h' = e'\sin\, v', \quad l' = e'\cos v',$$

are known at the epoch, and equations (132) give

$$g^2 - g\left\{\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}\right\}(4.5) = \frac{m\sqrt{a}}{m'\sqrt{a'}}\left\{\boxed{4.5}^2 - (4.5)^2\right\};$$

whence[28]

$$g_{/} = 3''.5851; \qquad g = 21''.9905;$$

$$N = -0.01715; \qquad N_{/} = 0.04321; \qquad N' = 0.04877;$$

$$N_{/}' = 0.03532; \qquad x_{/} = 210^\circ 16' \ 40''; \qquad x = 306^\circ \ 34' \ 40'';$$

and equation (135) gives

$$e = \sqrt{h^2 + l^2},$$

or

$$e = 0.04649\sqrt{1 + 0.68592\cos\left(83^\circ \ 42' - t\times 18''.4054\right)}$$

for the eccentricity of Jupiter's orbit; and

$$e' = 0.06021\sqrt{1 - 0.95009\cos\left(83^\circ \ 42' - t\times 18''.4054\right)}$$

for that of Saturn for any number t of Julian years after the epoch.

The longitudes of the perihelia are found from the value of $\tan v$ in article 495. The greatest deviation of these from their mean place will happen when

$$\cos\left\{(g - g_{/})t + x - x_{/}\right\} = -\frac{gN'^2 + g_{/}N_{/}'^2}{(g + g_{/})N'.N_{/}'}.$$

If this fraction be less than unity, the perihelia will librate like the nodes about a mean position, if not, they will move continually in one direction. In the case of Jupiter and Saturn $gN'^2 + g_{/}N_{/}'^2$ is greater than $(g + g_{/})N'.N_{/}';$ so that the perihelia go on for ever in one direction.

The period in which the eccentricities accomplish their changes is

$$\frac{360^\circ}{g - g_{/}} = \frac{360^\circ}{18''.4054} = 70{,}414 \text{ Julian years}.$$

The greatest and least values of the eccentricities are expressed by

$$N' \pm N'_{/} \text{ and } N \pm N_{/}.$$

For Saturn these are

$$0.08409 \text{ and } 0.01345,$$

and for Jupiter

$$0.06036 \text{ and } 0.02606;$$

the maximum of one planet corresponding to the minimum of the other.

The numerical values of the perturbations of the other planets will be found in the *Mécanique Céleste*; it is therefore only necessary to observe the circumstances that are peculiar to each planet.

Mercury

635. The motions of Mercury are less disturbed than those of any other body, on account of his proximity to the sun, his greatest elongation not exceeding $28^{\circ}.8$. His periodic inequalities are caused by Venus, the Earth, Jupiter, and Saturn; those from Saturn are very small, and Mars only affects the elements of his orbit.

The secular variations in the elements of Mercury's orbit were in the beginning of the year 1801, in the eccentricity

$$0.000003867;$$

secular and sidereal variation in the longitude of the perihelion,

$$9'43''.5;$$

secular and sidereal variation in the longitude of the node,

$$-13'2'';$$

secular variation of the inclination of the orbit on the true ecliptic,

$$19''.8.$$

636. Mercury sometimes appears as a morning and sometimes as an evening star, and exhibits phases like the moon. He occasionally is seen to pass over the disc of the sun like a black spot: these transits are true annular eclipses of the sun, proving that Mercury is an opaque body shining only by reflected light. The recurrence of the transits of Mercury depends on his periodic time being nearly equal to four times that of the earth. This ratio can be expressed by several pairs of small whole numbers, so that if the planet be in conjunction with the sun while in one of his nodes, he will be in conjunction again at the same node, after the Earth and he have completed a certain number of revolutions. The periodic revolutions of the earth have the following ratios to those of Mercury:

Periods of the Earth	Periods of Mercury
7	29
13	54
33	137
&c.	&c.

Consequently transits of Mercury will happen at intervals of 7, 13, 33, &c. years.

Had the orbit of Mercury coincided with the plane of the ecliptic, there would have been a transit at each revolution; but in consequence of the inclination of his orbit, transits do not happen often; for when a transit takes place, the latitude of Mercury must be less than the apparent semi-diameter of the sun. The return of the transits are also irregular from the great eccentricity of the orbit, which makes the motion of Mercury very unequal; the retrograde motion of the nodes also prevents the planet from returning to the same latitude when it returns to the same conjunction. A transit of Mercury took place at the descending node in 1799, the next that will happen at that node will be in 1832.

Transits happened at the ascending node in the years 1802, 1815, and 1822.

The mean apparent diameter of Mercury is $6''.9$.

Venus

637. 'The Morning Star' is the only planet mentioned in the sacred writings, and has been the theme of the poet's song, from Hesiod[29] and Homer,[30] to the days of Milton.[31]

Venus is next to Mercury, and exhibits similar phenomena. Like him she is alternately an evening and a morning star, has phases, and when in her nodes, occasionally appears to pass over the sun's disc, though her transits are not so frequent as those of Mercury. The returns of the transits of Venus depend on five times the mean motion of the earth being nearly equal to three times that of Venus: this however cannot be expressed by pairs of small whole numbers as in the case of Mercury; therefore the transits of Venus do not happen so often. It appears from the ratio of the periodic time of Venus to that of the earth, that eight periods of the earth's revolution are nearly equal to thirteen periods of the revolution of Venus, and 235 periods of the earth are nearly equal to 382 of Venus; hence a transit of Venus may happen at the same node after an interval of eight years, but if it does not happen, it cannot take place again at the same node for 235 years. At present, the heliocentric longitude of Venus's ascending node is something less than 75°, and that of her descending node is about 164°. The earth, as seen from the sun, has nearly the former longitude in the beginning of December, and the latter in the beginning of June; hence the transits of Venus for ages to come will happen in December and June. Those of Mercury will take place in May and November.

Table of the Transits of Venus

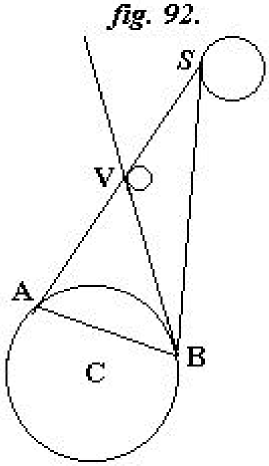
Year	
1631	6th December, ascending node.
1639	4th December, same.

1761	5th June, descending node.
1769	3rd June, same.
1874	8th December, ascending node.
1882	6th December, same.
2004	7th June, descending node.

The transits of Venus afford the most accurate method of finding the sun's parallax, and consequently his distance from the earth, from whence the true magnitude of the whole system is determined; for unless the actual distance of the sun were known, only the ratios of the magnitudes could have been ascertained.

638. The sun's parallax EmE′, fig. 65, which is the angle subtended at the sun by the earth's radius, can be found, if another angle EmE′, fig. 66, subtended by a chord EE′ lying between two known places on the earth's surface be known; that is, if the sun's parallax at any one altitude be known, his horizontal parallax may be determined, as it has been shown in article 329. However, the method employed in that number is not sufficiently accurate when applied to the sun, because in measuring the zenith distances, an error of three or four seconds might happen, which is immaterial in the case of the moon, whose parallax is nearly a degree, but an error of that magnitude in finding[32] the parallax of the sun, which is less than nine seconds, would render the results useless; hence, astronomers have endeavoured to compute the angle EmE′ instead of measuring it. Let AB, fig. 92, represent the equator, *S* and V the discs of the sun and Venus perpendicular to it: suppose them both to be moving in the equator, the motion of Venus retrograde, that of the sun direct. To a person at A, the internal contact, or total ingress of Venus on the sun commences, when to a spectator at B, the edge of Venus's disc is distant from the sun by the angle VB*S*. The difference between the times of total ingress as seen from B and A is the time of describing VB*S* by the approach of the sun and Venus to each other. Hence from the difference of the times, and the rate at which Venus and the sun approach each other, the angle VB*S* may be found, because the motions of both the sun and Venus are known. And sine VB*S* is to sine V*S*B, as Venus's distance from the sun to Venus's distance from the earth. But the ratio of Venus's distance from the sun to her distance from the earth is known, therefore the angle A*S*B is found, and C*S*B, the parallax of the sun may be computed, and from that his horizontal parallax; whence the distance of the sun from the earth may be determined in multiples of the terrestrial radius, or even in miles since the length of the radius is known. The computation of the transit is complicated chiefly on account of the inclination of Venus's orbit to the ecliptic, and the situations of the places of observation A and B being always at different distances from the equator. The investigation of this problem, and the computation of the parallax, will be found in Biot's and Woodhouse's *Astronomy*.[33]

fig. 92.



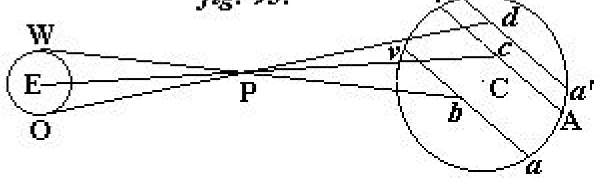
The times of internal contact can be observed with much greater accuracy than any angular distance can be measured, and on this depends the superiority of the preceding method of finding the parallax.

At inferior conjunction, the sun and Venus approach each other at the rate of 4″ in a minute; hence, if the time of contact be erroneous at each place of observation 4″ of time, the angle VB*S*, fig. 92, may be erroneous $\frac{4\times 8}{60}=\frac{8}{15}$ of a second, therefore the limit of the error in A*S*B is about $\frac{4}{15}$ of a second, and thus by the transit of Venus, an angle only $\frac{4}{15}$ of a second can be measured, a less quantity than can be determined by any other method.

639. The preceding method requires the difference of longitudes of the two places A and B to be accurately known, in order to compare the actual times of contact. In 1761 a transit of Venus was observed at the Cape of Good Hope, and at many places in Europe, the longitudes of all being well known: by comparing the observations the mean result determined the parallax to be $8''.47$; this is only an approximate value, but it was useful in obtaining the true value from the transit of 1769, which was observed at Wardhus in Lapland, and in Otaheite[34] in the southern hemisphere;[35] but as the longitude of the latter was unknown, astronomers avoided the difficulty by changing their method of calculation. In place of observing the ingress only, they observed the duration of the transit, and from the difference of duration at different places, they deduced the parallax.

Let P be Venus; E the earth, W Wardhus towards the north pole; O Otaheite towards the south; and VA the disc of the sun: then the true line of transit seen from E, the centre of the earth, would be VA, at W the transit would appear to be in the line *va*, and from O it would be seen in $v'a'$.

fig. 93.



If T be the true duration of the transit, or the time of describing VA, then the time of describing *va* nearer to the sun's centre, and therefore greater than VA, would be $\mathrm{T}+t$; whilst that of describing $v'a'$, which is farther from the centre, and therefore less than VA, would be $\mathrm{T}-t'$. The difference of the durations of the transits seen from O and W is $\mathrm{T}+t-(\mathrm{T}-t')=t+t'$, which is entirely the effect of parallax. With an approximate value of the parallax, t and t', the differences in the durations at W and O from what they would have been if observed at[36] E, the centre of the earth may be computed; then comparing the computed value of $t+t'$ with its observed value, the error in the assumed parallax will be found. With the parallax $8''.83$ it has been calculated that at Wardhus the duration was lengthened by $11'.16''.9$

And diminished at Otaheite by	$12'.10''$
Sum $t+t'$	$23'.26''.9$
But by observation	$23'.10''$
Difference	$16''.9$

Consequently the parallax $8''.45$ is less than that assumed; therefore to make the observed and computed differences of durations agree, the parallax must be $8''.72$. This does not differ much from what is given by the lunar theory $8''.6$, but an error recently detected by M. Bessel,[37] reduces it to $8''.575$. The transit commenced at Otaheite at half-past nine in the morning, and ended at half-past three in the afternoon.

640. Venus is by far the most brilliant and beautiful of the planets, but her splendour is variable. Her phases increase with her distance from the earth, and therefore she ought to become brighter as her disc enlarges; but the increase of the distance diminishes her lustre, since the intensity of light decreases proportionally to the square of the distance: there is, however, a mean position in which Venus is more brilliant than in any other; the interval of her returns to that position is about eight years, depending on the ratio of her periodic time to that of the earth. She is then visible to the naked eye during the day, but she is also visible in daylight every eighteen months though less distinctly.

The variations in the apparent diameter of Venus are very great; she is nearest the earth in her transit; her apparent diameter is then $16''.904$. M. Arago[38] has found its mean value to be $16^{\circ}.904$.

Shröeter, by observing the horns of Venus, determined her rotation about an axis, considerably inclined to the plane of the ecliptic, to be performed in $23^{h}\ 21'$; he discovered also very high mountains on her surface.

641. Venus is too near the sun to be very irregular in her motions, her greatest elongation not exceeding $47^{\circ}\ 7'$. In 1801, the secular variation in the eccentricity of her orbit was 0.000062711.

In the longitude of the perihelion,	$4'28''$
In the longitude of the ascending node,	$-31'10''$
In the inclination on the true ecliptic,	$4''.5$

The Earth

642. Uranus is too distant to have a sensible influence on the earth. Besides the disturbances occasioned by the other planets, there are some inequalities produced by the moon which are to be found in article 498.

It will be shown in the theory of the moon, that if $U-\Omega$ be her distance from her ascending node, the greatest inequality in her latitude is

$$18542''.8\sin(U-\Omega),$$

and if $S=18542''.8,$ the inequality (195) in the earth's latitude is

$$-\frac{m}{E}\cdot\frac{r}{\overline{r}}\cdot 18542''.8\sin(U-\Omega).$$

In order to compute the inequalities occasioned by the moon, it is requisite to know the ratio of the mass of the moon to that of the earth. The theory of the tides shows that the action of the moon in raising the waters of the ocean is 2.35333 times greater than that of the sun. The action of the moon on the earth, resolved in the direction r, is $\frac{E+m}{r^3}$; and the action of the sun, according to his radius vector $\overline{r}$, is $\frac{S}{\overline{r}^3}$: S and m being the masses of the sun and moon; hence

$$\frac{E+m}{r^3} = 2.35333 \cdot \frac{S}{r^3}.$$

By the theory of central forces,

$$\frac{E+m}{r^3} = n_/^2, \text{ and } \frac{S}{r^3} = n^2;$$

n and $n_/$ being the mean motions of the earth and moon; whence

$$\frac{m}{E+m} = 2.35333 \cdot \frac{n^2}{n_/^2}.$$

By observation,

$$\frac{n}{n_/} = 0.0748301;$$

hence

$$\frac{m}{E+m} = \frac{1}{75.928};$$

and if the mass of the earth be taken as the unit, the mass of the moon is

$$\frac{m}{E} = m = \frac{1}{75} \text{ nearly}.$$

Again, the ratio of the earth's distance from the sun to its distance from the moon is equal to the horizontal parallax of the sun, divided by the mean horizontal parallax of the moon, as will appear by considering that, as the parallax of both the sun and moon is very small, the arc may be taken for the sine, and the mean horizontal parallax of the moon is then the mean terrestrial radius divided by the mean distance of the moon from the earth; and the solar parallax is equal to the same terrestrial radius divided by the mean distance of the earth from the sun. The parallax of the sun is known, by observation, to be $8''.575,$ that of the moon is $3454''.16;$ hence the ratio of the distances is

$$\frac{8''.575}{3454''.16}.$$

With these data, the coefficients are

$$dv = -6''.8274 \sin(U - v),$$
$$dr = -0.0000331 \cos(U - v),$$
$$ds = -0''.61377 \sin(U - \Omega).$$

643. The inequality caused by the moon in the earth's radius vector is small; the mass of the moon being only $\frac{1}{75}$ part of that of the earth, the distance of the common centre of gravity of the earth and moon from the centre of the former must be less than the semidiameter, that is, it must be within the mass of the earth, and therefore the inequality in the earth's place must be less than $8''.575$, the sun's horizontal parallax.

644. The inequality produced by the moon in the earth's longitude is the lunar equation of the tables of the sun; it is of much importance for correcting the value of the mass of the moon. Its coefficient being computed with a value of the mass of the moon determined from the theory of the tides, compared with the coefficient of the same inequality determined by observation, will give the error in the mass of the moon, supposing the parallax of the sun and moon to be correct.

645. The irregularities communicated to the earth by the moon and planets are referred to the sun by observers on the earth's surface; therefore the sun appears to have a motion in longitude, by which he alternately advances before, and falls behind the point that describes the elliptical orbit in the heavens. In like manner be seems alternately to ascend above the plane of the ecliptic, and to descend below it by the disturbance in latitude. The perturbations in latitude, by the action of the planets, are computed from (160), and are[39]

$$ds''=\left\{\begin{array}{l}+0''.991803\sin\left(2n''t-n't+2\epsilon''-\epsilon'-q'\right)\\+0''.234256\sin\left(4n''t-3n't+4\epsilon''-3\epsilon'-q'\right)\\+0''.164703\sin\left(2n^{\text{iv}}t-n''t+2\epsilon^{iv}-\epsilon''-q^{\text{iv}}\right)\end{array}\right\};$$

this, added to

$$-0.61377\sin\left(U-\Omega\right),$$

is the whole periodic disturbance in the earth's motion in latitude, taken with a different sign. It affects the obliquity of the ecliptic, determined from the observations of the altitude of the sun in the solstices; it also has an influence on the time of the equinoxes, determined from observations of the sun at that period, as well as on the right ascensions and declinations of the fixed stars, determined by comparison with the sun; for it is clear that any inequalities in the motion of the earth will be referred to the observations made at its surface.

Considering the great accuracy of modern observations, these circumstances must be attended to. It is easy to see that this variation in the sun's latitude will increase his apparent declination by

$$-\frac{ds''.\cos\{\text{obliquity of ecliptic}\}}{\cos\{\text{declination of sun}\}};$$

and his apparent right ascension by

$$\frac{ds''.\sin\{\text{obliquity of ecliptic}\}.\cos\{\text{sun's R.A.}\}}{\cos\{\text{declination of sun}\}}$$

The observed right ascensions and declinations of the sun must therefore be diminished by these quantities, in order to have those that would be observed if the sun never left the plane of the ecliptic.

Secular Inequalities in the Terrestrial Orbit

646. The eccentricity and place of the perihelion of the terrestrial orbit may be determined with sufficient accuracy for 1000 or 1200 years before and after the epoch 1750, from

$$e = 2\bar{e} - 0''.187638 \times t - 0''000006721 \times t^2, \text{ and}$$
$$v = \bar{v} + 11''.949588 \times t + 0''.000079522 \times t^2,$$

$\bar{e}$ and $\bar{v}$ are the eccentricity and longitude of the perihelion at the epoch.

The secular diminution of the eccentricity is $18''.79$, about 3,914 miles, in reality an exceedingly small fraction in astronomy, though it appears so great in terrestrial measures. Were the diminution uniform, which there is no reason to believe, the earth's orbit would become a circle in 36,300 years; its variation has a great influence on the motions of the moon.

The longitude of the perihelion increases annually at the rate of $11''.9496$, so that it accomplishes a sidereal revolution in 109,758 years.

647. A remarkable period in astronomy was that in which the greater axis of the terrestrial orbit coincided with the line of the equinoxes, then the true equinox coincided with the mean. This occurred 4,084 years before the epoch in which chronologists place the creation of man;[40] at that time the solar perigee coincided with the equinox of spring. This however is but an approximate value, on account of the masses of the planets and the doubts as to the exact value of precession; the error may therefore be 80 years, which is not much in such a quantity.

Another remarkable astronomical period was, when the greater axis of the terrestrial orbit was perpendicular to the line of equinoxes; it was then that the true and mean solstice were united; this coincidence took place in the year 1248 of the Christian era. It is evident that these two periods depend on the direct motion of the perihelion and precession of the equinoxes conjointly.

648. The position of the ecliptic is changed by the reciprocal action of the planets on one another, and on the earth, each of them producing a retrograde motion in the intersection of the plane of its own orbit with the plane of the ecliptic. This action also changes the position of the plane of the ecliptic, with regard to itself, a change that may be determined from the values of p and q by formulae (138), or rather from

$$p = +0''.0767209 \times t + 0''.000021555 \times t^2,$$
$$q = -0''.5009545 \times t + 0''.000067473 \times t^2.$$

These will give the variation of the ecliptic, with regard to its fixed position in 1750, for 1000 or 1200 years, before and after that epoch.

This change in the ecliptic alters its position with regard to the earth's equator; but as the formulae in article 498 are periodic, these two planes never have and never will coincide. It occasions also a small motion in the equinoxes of about $0''.0846$ annually. Both of these variations are entirely independent of the form of the earth, and would be the same were it a sphere. However, the action of the sun and moon on the protuberant matter at the earth's equator is the cause of the precession of the equinoxes, or of that slow angular motion by which the intersection of the equator and ecliptic goes backward at the rate of $50''.34$ annually, so that the pole of the equator describes a circle round the pole of the ecliptic in the space of 25,748 years. This motion is diminished by the very small secular inequality $0''.0846$, arising from the action of the planets on the ecliptic. The formulae for computing the obliquity of the ecliptic and precession of the equinoxes depend on the rotation of the earth.

Mars

649. Mars is troubled by all the planets except Mercury. Jupiter alone affects the latitude of Mars. The secular variations in the elements of his orbit were, in 1801, as follow:

In the eccentricity.	0.000090176
In the longitude of the perihelion	$26''.22$
In the inclination on the true ecliptic	$1''.5$
In the longitude of the ascending node . . .	$-38'\,48''$

The eccentricity is diminishing.

The greatest elongation of Mars is[41] $126^\circ.8$. By spots on his surface it appears that he rotates in one day about an axis that is inclined to the plane of the ecliptic at an angle of $59^\circ.697$. His equatorial is to his polar diameter in the ratio of 194 to 189; his apparent diameter subtends an angle of $6''.29$, at his mean distance, and of $18''.28$ at his greatest distance, when his parallax is nearly twice that of the sun. The disc of Mars is occasionally gibbous. Spots near his poles that augment or diminish according as they are exposed to the sun, give the idea of masses of ice.

The New Planets

650. The orbits of Vesta, Juno, Ceres and Pallas are situate between those of Mars and Jupiter. Ceres was discovered by Piazzi, at Palermo,[42] on the first day of the present century; Pallas was discovered by Olbers,[43] in 1802; Juno in 1803, by Harding; and Vesta in 1807, by Olbers. These bodies are nearly at equal distances from the sun, their periodic times are therefore nearly the same. The eccentricities of the orbits of Juno and Vesta, and the position of their nodes are nearly the same.

These small planets are much disturbed by the proximity and vast magnitude of Jupiter and Saturn, and the series which determine their perturbations converge slowly, on account of the greatness of the eccentricities and inclinations of their orbits. The inclination of the old planets is so small, that they are all contained within the zodiac, which extends 8° on each side of the ecliptic, but those of the new planets very much exceed these limits. They are invisible to the naked eye, and so minute that their apparent diameters have not yet been measured. Sir William Herschel[44] estimated that they cannot amount to the fourth of a second, which would make the real diameter less than 65 miles. However, Juno, the largest of these asteroids, is supposed to have a real diameter of about 200 miles.

Jupiter

651. Jupiter is the largest planet in the system, and with his four moons exhibits one of the most splendid spectacles in the heavens. His form is that of an oblate spheroid whose polar diameter is $35''.65$, and his equatorial $= 38''.44$; he rotates in 9 hours 56 minutes about an axis nearly perpendicular to the plane of the ecliptic. The circumference of Jupiter's equator is about eleven times greater than that of the earth, and as the time of his rotation is to that of the earth as 1 to 0.414, it follows that during the time a point of the terrestrial equator describes 1°, a point in the equator of Jupiter moves through $2^{\circ}.41$; but these degrees are longer than the terrestrial degrees in the ratio of 11 to 1, consequently each point in Jupiter's equator moves 26 times faster than a point in the equator of the earth. In the beginning of 1801 the secular variations of his orbit were,

In the eccentricity.	0.00015935
In the longitude of the perihelion	$11'.4''$
In the inclination on the true ecliptic	$23''$
In the longitude of the ascending node . . .	$-26'\,17''$

Saturn

652. Viewed through a telescope Saturn is even more interesting than Jupiter: he is surrounded by a ring concentric with himself, and of the same or even greater brilliancy; the ring exhibits a variety of appearances according to the position of the planet with regard to the sun and earth, but is generally of an elliptical form: at times it is invisible to common observation, and can only be seen with superior instruments; this happens when the plane of the ring either passes through the centre of the sun or of the earth, for its edge, which is very thin, is then directed to the eye. On the 29th September, 1832, the plane of the ring will pass through the centre of the earth, and will be seen with a very high magnifying power like a line across the disc of the planet. On the 1st December of the same year, the plane of the ring will pass through the sun. Professor Struve[45] has discovered that the rings are not concentric with the planet. The interval between the outer edge of the globe and the outer edge of the ring on one side is $11''.037$, and on the other side the interval is $11''.288$, consequently there is an eccentricity of the globe in the ring of $0''.215$. In 1825 the ring of Saturn attained its greatest ellipticity; the proportion of the major to the minor axis was then as 1000 to 498, the minor being nearly half the major. Stars have been observed between the planet and his ring. It is divided into two parts by a dark concentric band, so that there are really two rings, perhaps more. These revolve about the planet on an axis perpendicular to their plane in about $10^h\ 29^m\ 17^s$, the same time with the planet.

The form of Saturn is very peculiar. He has four points of greatest curvature, the diameters passing through these are the greatest; the equatorial diameter is the next in size, and the polar the least; these are in the ratio of 36, 35, and 32. Besides the rings, Saturn is attended by seven satellites which reciprocally reflect the sun's rays on each other and on the planet. The rings and moons illuminate the nights of Saturn; the moons and Saturn enlighten the rings, and the planet and rings reflect the sun's beams on the satellites when they are deprived of them in their conjunctions. The rings reflect more light than the planet. Sir William Herschel observed, that with a magnifying power of 570, the colour of Saturn was yellowish, whilst that of the rings was pure white. Saturn has several belts parallel to his equator: changes have been observed in the colour of these and in the brightness of the poles, according as they are turned to or from the sun, probably occasioned by the melting of the snows. Saturn's motions are disturbed by Jupiter and Uranus alone; the secular variations in the elements of his orbit were as follows, in the beginning of 1801.

In the eccentricity. 0.000312402
In the longitude of the perihelion . . . . $32'.17''$
In the inclination on the true ecliptic . . . . . . $15'\ 5''$
In the longitude of the ascending node . . . . $-37'\ 54''$

Uranus, or the Georgium Sidus

653. This planet was discovered by Sir William Hershel, in 1781. The period of his sidereal revolution is 30,687 days. If we judge of the distance of the planet by the slowness of its motion, it must be on the very confines of the solar system; its greatest elongation is $103^\circ.5$, and its apparent diameter $4''$: it is accompanied by six satellites, only visible with the best telescopes. The only sensible perturbations in the motions of this planet arise from the action of Jupiter and Saturn; the secular variations in the elements of its orbit were, in 1801, as follow:

In the eccentricity. 0.000025072
In the longitude of the perihelion . . . . $4'$
In the inclination on the true ecliptic . . . . . . $3''.7$
In the longitude of the ascending node . . . . $-59'\ 57''$

The rotation of Uranus[46] has not been determined.

654. It is remarkable that the rotation of the celestial bodies is from west to east, like their revolutions; and that Mercury, Venus, the Earth, and Mars, accomplish their rotations in about twenty-four hours, while Jupiter and Saturn perform theirs in $\frac{4}{10}$ of a day.

On the Atmosphere of the Planets

655. Spots and belts are observed on the discs of some of the planets varying irregularly in their position, which shows that they are surrounded by an atmosphere; these spots appear like clouds driven by the winds, especially in Jupiter. The existence of an atmosphere round Venus is indicated by the progressive diffusion of the sun's rays over her disc. Schroëter measured the extension of light beyond the semicircle when she appeared like a thin crescent, and found the zone that was illuminated by twilight to be at least four degrees in breadth, whence be inferred that her atmosphere, must be much more dense than that of the earth. A small star hid by Mars was observed to become fainter before its appulse[47] to the body of the planet, which must have been occasioned by his atmosphere. Saturn and his rings are surrounded by a dense atmosphere, the refraction of which may account for the irregularity apparent in his form: his seventh satellite has been observed to hang on his disc more than $20'$ before its occultation, giving by computation a refraction of two seconds, a result confirmed by observation of the other satellites. An atmosphere so dense must have the effect of preventing the radiation of the heat from the surface of the planet, and consequently of mitigating the intensity of cold that would otherwise prevail, owing to his vast distance from the sun. Schroëter observed a small twilight in the moon, such as would be occasioned by an atmosphere capable of reflecting the sun's rays at the height of about a mile. Had a dense atmosphere surrounded that satellite, it would have been discovered by the duration of the occultations of the fixed stars being less than it ought to be, because its refraction would have rendered the stars visible for a short time after they were actually behind the moon, in the same manner as the refraction of the earth's atmosphere enables us to see celestial objects for some minutes after they have sunk below our horizon, and after they have risen above it, or distant objects hid by the curvature of the earth. A friend of the author's was astonished one day on the plain of Hindostan,[48] to behold the chain of the Himala[49] mountains suddenly start into view after a heavy shower of rain in hot weather.

The Bishop of Cloyne[50] says, that the duration of the occultations of stars by the moon is never lessened $8''$ of time, so that the horizontal refraction at the moon must be less than $2''$: if therefore a lunar atmosphere exists, it must be 1000 times rarer than the atmosphere at the surface of the earth, where the horizontal refraction is nearly $2000''$. Possibly the moon's atmosphere may have been withdrawn from it by the attraction of the earth. The radiation of the heat occasioned by the sun's rays must be rapid and constant, and must cause intense cold and sterility in that cheerless satellite.

The Sun

656. The sun viewed with a telescope, presents the appearance of an enormous globe of fire, frequently in a state of violent agitation or ebullition; black spots of irregular form rarely visible to the naked eye sometimes pass over his disc, moving from east to west, in the space of nearly fourteen days: one was measured by Sir W. Herschel in the year 1779, of the breadth of 30,000 miles. A spot is surrounded by a penumbra, and that by a margin of light, more brilliant than that of the sun. A spot when first seen on the eastern edge, appears like a line, progressively extending in breadth till it reaches the middle, when it begins to contract, and ultimately disappears at the western edge: in some rare instances, spots re-appear on the east side; and are even permanent for two or three revolutions, but they generally change their aspect in a few days, and disappear: sometimes several small spots unite into a large one, as a large one separates into smaller ones which soon vanish.

The paths of the spots are observed to be rectilinear in the beginning of June and December, and to cut the ecliptic at in angle of $7^\circ\ 20'$. Between the first and second of these periods, the lines described by the spots are convex towards the north, and acquire their maximum curvature about the middle of that time. In the other half year the paths of the spots are convex towards the south, and go through the same changes. From these appearances it has been concluded, that the spots are opaque bodies attached to the surface of the sun, and that the sun rotates about an axis, inclined at an angle of $7^\circ\ 20'$ to the axis of the ecliptic. The apparent revolution of a spot is accomplished in twenty-seven days; but during that time, the spot has done more, having gone through a revolution, together with an arc equal to that described by the sun in his orbit in the same time, which reduces the time of the sun's rotation to $25^d\ 9^m\ 35^s$.

These phenomena induced Sir W. Herschel to suppose the sun to be a solid dark nucleus, surrounded by a vast atmosphere, almost always filled with luminous clouds, occasionally opening and discovering the dark mass within. The speculations of Laplace were different: he imagined the solar orb to be a mass of fire, and that the violent effervescences and explosions seen on its surface are occasioned by the eruption of elastic fluids formed in its interior, and that the spots are enormous caverns, like the craters of our volcanoes.

Light is more intense in the centre of the sun's disc than at the edges, although, from his spheroidal form, the edges exhibit a greater surface under the sane angle than the centre does, and therefore might be expected to be more luminous. The fact may be accounted for, by supposing the existence of a dense atmosphere absorbing the rays which have to penetrate a greater extent of it at the edges than at the centre; and accordingly, it appears by Bouguer's[51] observations on the moon, which has little or no atmosphere, that it is more brilliant at the edges than in the centre.

657. A phenomenon denominated the zodiacal light, from its being seen only in that zone, is somehow connected with the rotation of the sun. It is observed before sunrise and after sunset, and is a luminous appearance, in some degree similar to the milky way, though not so bright, in the form of an inverted cone with the base towards the sun, its axis inclined to the horizon, and only inclined to the plane of the ecliptic at an angle of 7°; so that it is perpendicular to the axis of the sun's rotation. Its length from the sun to its vertex varies from 45° to 120°. It is seen under the most favorable circumstances after sunset in the beginning of March: its apex extends towards Aldebaran,[52] making an angle of 64° with the horizon. The zodiacal light varies in brilliancy in different years.

It was discovered by Cassini[53] in 1682, but had probably been seen before that time. It was observed in great splendour at Paris on the 16th of February, 1769.

658. The elliptical motion of the planets is occasioned by the action of the sun; but by the law of reaction, the planets must disturb the sun, for the invariable point to which they gravitate is not the centre of the sun, but the centre of gravity of the system; the quantity of motion in the sun in one direction must therefore be equal to that of all the planets in a contrary direction. The sun thus describes an orbit about the centre of gravity of the system, which is a very complicated curve, because it results from the action of a system of bodies, perpetually changing their relative positions; it is such however as to furnish a centrifugal force with regard to each planet, sufficient to counteract the gravitation towards it.

Newton[54] has shown that the diameter of the sun is nearly equal to 0.009 of the radius of the earth's orbit. If all the great planets of the system were in a straight line with the sun, and on the same side of him, the centre of the sun would be nearly the farthest possible from the common centre of gravity of the whole; yet it is found by computation, that the distance is not more than 0.0085 of the radius vector of the earth; so that the centre of the sun is never distant from the centre of gravity of the system by as much as his own diameter.

Influence of the Fixed Stars in disturbing the Solar System

659. It is impossible to estimate the effects of comets in disturbing the solar system, on account of our ignorance of the elements of their orbits, and even of the existence of such as have a great perihelion distance, which nevertheless may trouble the planetary motions; but there is every reason to believe that their masses are too small to produce a sensible influence; the effect of the fixed stars may, however, be determined.

Let m' be the mass of it fixed star, x', y', z', its co-ordinates referred to the centre of gravity of the sun, and r' its distance from that point. Also let x, y, z, be the co-ordinates of a planet m, and r its radius vector; then the disturbing influence of the star is[55]

$$R = \frac{m'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} - \frac{m'(xx'+yy'+zz')}{r'^3};$$

or [56]

$$R = +\frac{m'}{r'} - \frac{m'r^2}{2r'^3} + \tfrac{3}{2}m'\frac{\left((xx'+yy'+zz') - \tfrac{1}{2}r^2\right)^2}{r'^5} + \&\text{c.}$$

when developed according to the powers of r'. The fixed plane being the orbit of m at the epoch, then

$$x = r\cos v, \quad y = r\sin v, \quad z = rs,$$

let l be the latitude of the fixed star, and u its longitude, then

$$x' = r'\cos l\cos u, \quad y' = r'\cos l\sin u, \quad z' = r'\sin l;$$

and if all the powers of r' above the cube be omitted, it will be found that

$$R = +\frac{m'}{r'} - \frac{m'r^2}{4r'^3}\left\{2 - 3\cos^2 l - 3\cos^2 l\,.\cos(2v-2u) - 6s\,.\sin 2l\,.\cos(v-u)\right\}.$$

But neglecting s, the substitution of this in equation (155) gives

$$\frac{dr}{a} = -\frac{m' a^3 nt}{r'^3}\left\{\left(1-\tfrac{3}{2}\cos^2 l\right)e\sin\left(v-v\ \right)-\tfrac{3}{4}\cos^2 l\,.\,e\,.\sin\left(v+v\ -2u\right)\right\}.$$

But

$$r = a\left(1+e\cos\left(v-v\ \right)\right);$$

whence

$$\frac{dr}{a} = de\cos\left(v-v\ \right)+edv\ .\sin\left(v-v\ \right);$$

and comparing the two values of $\frac{dr}{a}$, there will be found

$$dv\ = -\frac{m' a^3}{r'^3 e}\cdot nt\left\{1-\tfrac{3}{2}\cos^2 l-\tfrac{3}{4}\cos^2 l\cos\left(2v\ -2u\right)\right\}$$

$$de = \frac{3m' a^3}{4r'^3}\cdot\cos^2 lnt\ .e\,.\sin\left(2v\ -2u\right).$$

Whence it appears, that the star occasions secular variations in the eccentricity and longitude of the perihelion of m, but these variations are incomparably less than those caused by the planets. For if m be the earth, the distance of the star from the centre of the sun cannot be less than 100,000 times the mean distance of the earth from the sun, because the annual parallax of the nearest fixed star is less than $1''$; therefore assuming $r' = 100{,}000\,.\,a$ the coefficient $\frac{m' a^3}{r'^3}nt$ does not exceed $0''.0000000013\,.\,m't$, t being any number of Julian years. This quantity is incomparably less than the corresponding variation in the eccentricity of the earth's orbit, arising from the action of the planets, which is

$$-0''.0938191\,.\,t,$$

unless the mass m' of the fixed stars be much greater than what is probable. Whence it may be concluded that, the attraction of the fixed stars has no sensible influence on the form of the planetary orbits; and it may be easily proved, that the positions of the orbits are also uninfluenced.

Disturbing Effect of the Fixed Stars on the Mean Motions of the Planets

660. The part of equation (156) that depends on R, when $m = 1$, is

$$d\ .dz\ = -3a\int ndt\cdot \mathrm{d}R-2a\cdot ndt\cdot r\left(\frac{dR}{dr}\right).$$

The preceding value of R gives

$$d\,.dz = +\frac{m'a^3}{r'^3}ndt\left(2-3\cos^2 l\right)-\frac{6m'a^3}{r'^3}\cdot s\,.\sin 2l\,.\cos\left(v-u\right)$$
$$-\frac{9}{2}\cdot m'\,.\,a^3\,.\,ndt\int d\cdot\frac{s\,.\sin 2l}{r^3}\cdot\cos\left(v-u\right),$$

which is the whole variation in the mean motion of m from the action of the fixed stars. The parts will be examined separately.

Let r'' and l' be the distance and latitude of the star at the epoch 1750, and let it be assumed, that these quantities diminish annually by a and b, then t being any indefinite time, r' and l become

$$r' = r''\left(1-at\right),\quad l = l'\left(1-bt\right)$$

whence the first term of $d\,.dz$ becomes

$$d\,.dz = \frac{3m'\,.\,a^3}{r''^3}\left(1-\tfrac{3}{2}\cos^2 l'\right)ant^2-\frac{3m'\,.\,a^3}{r''^3}\cdot\sin 2l'\,.\,b\,.\,nt^2\,.$$

We know nothing of the changes in the distance of the fixed stars; but with regard to the earth, they may be assumed to vary $0''.324$ annually in latitude; hence[57]

$$b = 0''.324,\quad r'' = 100{,}000a,$$

so that $\frac{m'a^3}{r'^3}\cdot b\,.\,nt^2$ becomes

$$\frac{m't^2\,.\,2''.0357}{10^{15}}.$$

a quantity inappreciable from the earliest observations.

With regard to the terms in s,

$$s = t\cdot\frac{dp}{dt}\sin v - t\cdot\frac{dq}{dt}\cos v;$$

consequently, rejecting the periodic part,

$$d\cdot\frac{s\,.\sin 2l}{r'^3}\cdot\cos\left(v-u\right)=\frac{\sin 2l}{2r'^3}\left\{\frac{dp}{dt}\cdot\sin u-\frac{dq}{dt}\cdot\cos u\right\},$$

so that

$$d\,.dz = -\frac{21}{4}\cdot\frac{m'a^3}{r'^3}\cdot n\,.\,tdt\,.\sin 2l\left\{\frac{dp}{dt}\sin u - \frac{dq}{dt}\cos u\right\};$$

the integral of which is

$$dz = -\frac{21}{8}\cdot\frac{m'a^3}{r'^3}\cdot nt^2\,.\sin 2l\left\{\frac{dp}{dt}\sin u - \frac{dq}{dt}\cos u\right\}.$$

But with regard to the earth

$$p = +0''.076721\times t + 0''.000021555\times t^2$$
$$q = -0''.50096\times t + 0''.0000067474\times t^2.$$

If these quantities be substituted, it will be found that the secular inequalities in the mean motion of the earth are quite insensible; the earliest records also prove them to be so. The same results will be obtained for the most distant planets, whence it may be concluded that the fixed stars are too remote to affect the solar system.

Construction of Astronomical Tables

661. The motion of a planet in longitude consists of three parts, of the mean or circular motion; of a correction depending on the eccentricity, which is the equation of the centre; and of the periodic inequalities.

In the construction of tables, the mean longitude of the body, and the mean longitude of the aphelion, or perihelion, are determined in degrees, minutes, seconds, and tenths, at the instant assumed as the origin of the tables. These initial values are generally computed for the beginning of each year, and are called the epoch of the tables; from them subsequent values are deduced at convenient intervals, by adding the daily increments. These intervals are longer or shorter according to the motion of the body, or its importance, and the intermediate values are found by simple proportion, or by tables of proportional parts. The mean anomaly is given by the tables, since it is the difference between the mean longitudes of the body and of the aphelion.

The tables of the equation of the centre, and of the mean longitude of the aphelion, give these quantities for each degree of mean anomaly. To these are added tables of the periodic inequalities in longitude, and of the secular inequalities in the eccentricity and longitude of the aphelion. From these tables the true longitude of the body may be known at any instant, by applying the corrections to the mean longitude.

The radius vector consists of three parts,–of a mean value, which is equal to half the greater axis of the orbit; of the elliptical variations, and of its periodic inequalities. The two latter are given in the tables for every degree of mean anomaly. The latitude is computed in terms of the mean anomaly at stated intervals: besides these, the mean longitude of the ascending node and the inclination of the orbit at the beginning of each year, and the secular inequalities of these two quantities are given. Thus the mean motions are given, and the true motions are found by applying the inequalities, the numerical values of which are called equations: for, in astronomy, an equation signifies the quantities that must be added or taken from the mean results, to make them equal to the true results.

The mean motion and equation of the centre are computed from Kepler's problem; the motions of the nodes and perihelia, the secular inequalities of the elements, and the periodic inequalities, are computed from the formulae determined by the problem of three bodies.

Method of correcting Errors in the Tables

662. As astronomical tables are computed from analytical formulae, determined on the principles of universal gravitation, no error can arise from that source; but the elements of the orbit, though determined with great accuracy by numerous observations, will lead to errors, because each element is found separately; whereas these quantities are so connected with each other, that a perfectly correct value of one, cannot be determined independently of the others. For example, the expressions in articles[58] 477-479 show, that the eccentricity depends on the longitudes of the perihelia, and the longitude of the perihelion is given in terms of the eccentricities. A reciprocal connexion exists also between the inclination of the orbit and the longitude of the nodes. Hence, in an accurate determination of the elements, it is necessary to attend to this reciprocal connexion.

The tables are computed with the observed values of the elements; an error in one of the elements will affect every part of the tables, and will be perceived in the comparison [of][59] the place of the body derived from them, with its place determined by observation. Were the observation exact, the difference would be the true error of the tables; but as no observation is perfectly accurate, the comparison is made with 1000, or even many thousands of observations, so that their errors are compensated by their numbers.

The simultaneous correction is accomplished, by comparing a longitude of the body derived from observation, with the longitude corresponding to the same instant in the tables.

Suppose the tables of the sun to require correction, and let E represent the error of the tables, or the difference between the longitude of the tables and that deduced from observation, at that point of the orbit where his mean anomaly is 198°. There are three sources from whence this error may arise, namely, the mean longitude of the perigee, the greatest equation of the centre, and the epoch of the tables; for, if an error has been made in computing the initial longitude, it will affect every subsequent longitude. Now, as we do not know to which of these quantities to attribute the discrepancy, part of it is assumed to arise from each. Let P be the unknown error in the longitude of the perigee, e that in the greatest equation of the centre, and $\in$ that in the epoch. In order to determine these three errors, let us ascertain what effect would be produced on the place of the sun, where his mean anomaly is 198°, by an error of $60''$ in the longitude of the perigee. As the mean anomaly is estimated from perigee, a minute of change in the perigee will produce the change of one minute in the mean anomaly corresponding to each longitude; but the table of the equation of the centre shows that the change of $60''$ in the mean anomaly at that part of the orbit which corresponds to 198° produces an increment of $1''.88$ in the equation of the centre; and as that quantity is subtractive at that part of the orbit, the true longitude of the sun is diminished by $1''.88$; hence, if $60''$ produce a change of $1''.88$ in the true longitude, the error P will produce a change of

$$\frac{1''.88}{60''}P = 0''.3133P\,.$$

Again, if we suppose the greatest equation of the centre to be augmented by any arbitrary quantity as $17''.18$, it is easy to see by the tables that the equation of the centre at that point of the orbit where the mean anomaly is 198° is increased by $5''.1$; whence the true longitude is diminished by $5''.1$. Thus, if $17''.18$ produce a change of $5''.1$ in the true longitude, the error e will produce the change

$$-\frac{5''.1}{17''.18}e = -0''.2969e\,.$$

Hence the sum of the three errors is equal to E, the error of the tables

$$\in +0''.3133P - 0''.2969e = E\,.$$

This is called an equation of condition between the errors, because it expresses the condition that the sum of the errors must fulfil.

As there are three unknown quantities, three equations would be sufficient for their determination, if the observations were accurate; but as that is not the case, a great number of equations of condition must be formed from an equal number of observed longitudes, and they must be so combined by addition or subtraction, as to form others that are as favorable as possible for the determination of each element. For example, in finding the value of P before the other two, the numerous equations must be so combined, as to render the coefficient of P as great as possible; and the coefficients of e and $\in$ as small is may be; this may always be accomplished by changing the signs of all the equations, so as to have the terms containing P positive, and then adding them; for some of the other terms will be positive, and some negative, as they may chance to be; therefore the sum of their coefficients will be less than that of P.

Having determined this equation, in which P has the greatest coefficient possible, two others must be formed on the same principle, in which the coefficients of the other two errors must be respectively as great as possible, and from these three equations values of the three errors will be easily obtained, and their accuracy will be in proportion to the number of observations employed. These values are referred to the mean interval between the first and last observations, supposing them not to be separated by any great length of time, and that the mean motion is perfectly known. Were it not, as might happen in the case of the new planets, an additional error may be assumed to arise from this source, which may be determined in the same manner as the others. This method of correcting errors in astronomical tables was employed by Mayer, in computing tables of the moon, and is applicable to a variety of subjects.

663. The numerous equations of condition of the form

$$E = \in +0''.3133P + 0''.2969e,$$

may be combined in a different manner, used by Legendre,[60] called the principle of the least squares.

If the position of a point in space, is to be determined, and if a series of observations had given it the positions $n,\ n',\ n''$, &c., not differing much from each other, a mean place M must be found, which differs as little as possible from the observed positions $n,\ n',\ n''$, &c.: hence it must be so chosen that the sum of the squares of its distances from the points $n,\ n',\ n''$, &c., may be a minimum; that is,

$$(Mn)^2 + (Mn')^2 + (Mn'')^2 + \&c. = \text{minimum}\,.$$

A demonstration of this is given in Biot's Astronomy,[61] vol. ii.; but the rule for forming the equation of the minimum, with regard to one of the unknown errors, as P, is to multiply every term of all the equations of conditions by $0''.3133$, the coefficient of P, taken with its sign, and to add the products into one sum, which will be the equation required. If a similar equation be formed for each of the other errors, there will be as many equations of the first degree as errors; whence their numerical values may be found by elimination.

It is demonstrated by the Theory of Probabilities, that the greatest possible chance of correctness is to be obtained from the method of least squares; on that account it is to be preferred to the method of combination employed by Mayer, though it has the disadvantage of requiring more laborious computations.

The principle of least squares is a corollary that follows from a proposition of the Loci Plani, that the sum of the squares of the distances of any number of points from their centre of gravity is a minimum.

664. Three centuries have not elapsed since Copernicus[62] introduced the motions of the planets round the sun, into astronomical tables: about a century later Kepler[63] introduced the laws of elliptical motion, deduced from the observations of Tycho Brahe[64], which led Newton to the theory of universal gravitation. Since these brilliant discoveries, analytical science has enabled us to calculate the numerous inequalities of the planets, arising from their mutual attraction, and to construct tables with a degree of precision till then unknown. Errors existed formerly, amounting to many minutes; which are now reduced to a few seconds, a quantity so small, that a considerable part of it may perhaps be ascribed to inaccuracy in observation.

Notes

[1] *Numerical Values of the Perturbations.* This chapter title in the 1st edition *reads* "Numerical Values of the Perturbations of Jupiter."
[2] This *reads* "variation" in the 1st edition (published erratum).
[3] The first parenthesis in the last equation is rounded in the 1st edition.
[4] The fourth element $(d\overline{\Pi})$ *reads* $(d\Pi)$ in the 1st edition.
[5] The argument in the 17th term (next page) in the series for dv *reads* $\{3nt - 2nt + 3\in' - 2\in' - v\}$ in the 1st edition.
[6] The closing parenthesis in this equation is omitted in the 1st edition.
[7] The left hand side of the fifth equation *reads* $\sum .N\,.\sin(n't + nt + \in' + \in - L =$ in the 1st edition.
[8] In the 1st edition $n' = 43996''.6$ *reads* $n'' = 43996''.6$.
[9] This is incorrectly capitalized in the 1st edition.
[10] This expression is expressed in two parts containing eight and five terms respectively in the 1st edition.
[11] The + sign before the 9th term is omitted in the 1st edition.

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[12] The accent is omitted on the first instance of $\in'$ in the 1st edition.

[13] The left hand side term $4\frac{dP}{da}$ *reads* $\frac{dP}{da}$ in the 1st edition (published erratum).

[14] A comma is used at the end of this sentence in the 1st edition.

[15] The value $0''.0004491$ *reads* $0''.0054491$ in the 1st edition (published erratum).

[16] The value $0''.0004491$ *reads* $0''.0054491$ in the 1st edition (published erratum).

[17] This expression contains two published errata in the 1st edition: The last term *reads*:

$$+\frac{5m'}{2}\cdot Ke\,.\sin\left(5n't-2nt+5\in'-2\in-\nu+B\right).$$

and an erroneous third term *reads*:

$$+m'He\,.\sin\left(5n't-2nt+5\in'-2\in-\nu+B\right).$$

[18] See note 3, *Bk. II, Chap. IV.*

[19] Parentheses are omitted in the arguments of both cosines in the 1st edition.

[20] The first term *reads* $+0''.051737\,.\sin\left(n't-nt+\in'-\in\right)$ in the 1st edition (published erratum).

[21] Not capitalized in the 1st edition.

[22] $\tan f=N-N_{/}$ *reads* $\tan f=N-N'$ in the 1st edition (published erratum).

[23] *libratory*. An oscillation in the apparent aspect of a secondary body (as a planet or a satellite) as seen from the primary object around which it revolves. *Merriam-Webster Collegiate Dictionary*.

[24] See note 55, *Preliminary Dissertation*.

[25] In the 1st edition q and q' *read* 0 and $0'$ (published erratum).

[26] See note 64, *Preliminary Dissertation*.

[27] Spelled “immoveable” in the 1st edition.

[28] In the 1st edition g and $g_{/}$ are reversed (published erratum).

[29] Hesiod, c. 750 BC, poet, born in Ascra, Greece. One of the earliest known Greek poets, he is best known for two works, *Works and Days* and *Theogony*.

[30] Homer, c. 850 BC, Greek poet, author of the *Odyssey* and the *Iliad*.

[31] Milton, John, 1608-1674, poet, born in London, England. His major works were *L'Allegro* and *Il Penseroso* (1632), *Comus* (1633), *Lycidas* (1637), *Areopagitica* (1644),), *Paradise Lost* (1663), *Paradise Regained* (1671) and *Samson Agonistes* (1671).

[32] This word is omitted in the 1st edition (published erratum).

[33] Biot, Jean Baptiste, 1774-1862, *Traite elementaire d'astronomie physique par J.B. Biot; avec des additions relatives a l'astronomie nautique par M. de Rossel*, Paris: J. Klostermann, 1810-11 (see also note 17, *Bk. I*, *Chap. II*)

[34] *Otaheite*. Known now as Tahiti.

[35] Measurements made by Nevil Maskelyne, 1732-1811 (see note 55, *Preliminary Dissertation* and note 5, *Bk. II*, *Chap. IX*).

[36] This *reads* C, in the 1st edition (published erratum).

[37] Bessel, Friedrich Wilhelm, 1784-1846, mathematician and astronomer, born in Minden, Germany. He was appointed director of the observatory and professor at Königsberg in 1819. He catalogued stars, and predicted a planet beyond Uranus as well as the existence of dark stars. He was also the first person to measure stellar parallax (of the star 61 Cygni) in 1838. Bessel’s measured value of 0.314 seconds compares favorably with the modern value of 0.292 seconds. Struve (see note 29, *Preliminary Dissertation*) independently measured the parallax of the star Vega in 1839. Bessel later investigated the so-called Kepler's problem of heliocentricity. A series of mathematical functions bear Bessel’s name.

[38] Arago, (Dominique) François (Jean), 1786-1853, scientist, born in Estagel, France. In 1830 he became secretary to the Polytechnic Observatory where he had worked since the age of 17. His was active in areas of astronomy, magnetism, and optics. Arago is known also for his involvement in a dispute between U. Leverrier (see note 28, *Bk. II Foreword*) and John Adams (see note 39, *Bk. II Foreword*) over priority in the discovery of the planet Neptune. (see also note 48, *Bk. I Foreword*, and *Foreword to the second edition*.)

[39] The + sign on the second term is missing in the 1st edition, as is the closing parenthesis.

[40] A determination based on a chronology of biblical events placed the creation of man around 4000 BCE.

[41] This *reads* $126\overset{\circ}{.}8$ in the first edition.

[42] Piazzi, Giuseppe, 1746-1826, astronomer and theologian, born in Ponte di Valtellina, Italy. Piazzi was a Theatine monk and professor of theology and mathematics in Rome (1779) and Palermo (1780) where he established an observatory in 1789. He also measured the proper motion of the star 61 Cygni, the same star Bessel later used for his first measurement of stellar parallax (see note 37 above). Piazzi is known best for his discovery and naming of the first minor planet or asteroid, Ceres (see note 9, *Preliminary Dissertation*).

[43] Olbers, (Heinrich) Wilhelm (Matthäus), 1758-1840, physician and astronomer, born in Arbergen, Germany. He invented a method for calculating the velocity of falling stars. He also discovered the minor planets or asteroids Pallas (1802) and Vesta in 1807 (see note 9, *Preliminary Dissertation*) as well as five new comets (one bears his name). He theorized that a comet's tail was due to radiation pressure. Light pressure was demonstrated experimentally in the 20th century.

[44] See note 26.

[45] See note 29, *Preliminary Dissertation*.

[46] The 1st edition text *reads* Saturn *for* Uranus.

[47] *appulse*. In astronomy, the approach of any planet to a conjunction with the sun, or a star. *Webster's 1828 Dictionary*.

[48] *Hindostan*. 'The country of the Hindus,' India. In modern native parlance this word indicates distinctively India north of the Nerbudda, and exclusive of Bengal and Behar. *The Anglo-Indian Dictionary*.

[49] *Himala*. Himalayas *The Anglo-Indian Dictionary*.

[50] Berkeley, George, 1685-1753, Anglican bishop and philosopher, born at Dysert Castle, Kilkenny, Ireland. He studied at Trinity College, Dublin, where he remained, as fellow and tutor, until 1713. His most important works include: *Essay towards a New Theory of Vision* (1709), *A Treatise concerning the Principles of Human Knowledge* (1710), and *Three Dialogues between Hylas and Philonous* (1713). He became Bishop of Cloyne in 1734.

[51] Bouguer, Pierre, 1698-1758, physicist, born in Le Croisie, France. In 1735 he was sent with others to Peru to measure a degree of the meridian at the equator. His views on the intensity of light laid the foundation of photometry. In 1748 he invented the heliometer.

[52] *Aldebaran*. An orange binary star in the constellation Taurus, having a combined magnitude of 0.9.

[53] Cassini, Giovanni (Gian) Domenico, 1625-1712, astronomer, born in Perinaldo, Italy. He was a professor of astronomy at Bologna, and the first director of the observatory at Paris (1669). Cassini measured solar parallax, as well as the periods of Mars, Venus and Jupiter (by measuring the shadows of Jupiter's shadows as they passed between the planet and the sun). He was also first to document the zodiacal light and first to observe Saturn's four moons (Iapetus, Rhea, Tethys, and Dione) as well as the gap between two of Saturn's rings (Cassini's Division). Cassini's laws on lunar rotation were formulated in 1693.

[54] See note 1, *Preliminary Dissertation*.

[55] The numerator in the second term *reads* $m'(xx+yy+zz)$ in the 1st edition.

[56] A parenthesis in the numerator in the second term is missing and *reads* $(xx'+yy'+zz')-\frac{1}{2}r^2)^2$ in the 1st edition.

[57] The second value *reads* $r'=100.000\,a$ in the 1st edition.

[58] This *reads* "in page 261," in the 1st edition (original pagination).

[59] *[of]*. This word is corrupted in the 1st edition.

[60] Legendre, Adrien-Marie, 1752-1833, mathematician, born in Paris. Legendre was professor of mathematics at the École Militaire (1775-80), and professor at the Ecole Normale (1795). The method of least squares mentioned by Somerville was indeed devised by Legendre and appeared in an appendix to his *Nouvelles méthodes pour la détermination des orbites des comètes* in 1806. However Carl Gauss (see note 8, *Bk. II, Chap XIII*) claimed priority for the method in his *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium* in 1809 while acknowledging that it had appeared earlier in Legendre's work. At École Militaire Legendre taught with Pierre-Simon Laplace (see note 4, *Foreword to the Second Edition*) who is said to have appropriated some of Legendre's work with little credit to him. Legendre made important contributions to number theory and elliptic integrals (*Traité des fonctions elliptiques* in 1825-37). He studied the attraction of spheroids and ellipsoids. Using what are known now as Legendre functions he was able to determine the attraction of an ellipsoid at any exterior point.

[61] Op. Cit.

[62] See note 1, *Bk. II, Chap. I.*

[63] See note 3, *Preliminary Dissertation.*

[64] See note 6, *Bk. II, Chap. I.*