BOOK II

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CHAPTER X

THE THEORY OF JUPITER AND SATURN

571. BY comparing ancient with modern observations, Halley[1] discovered that the mean motion of Jupiter had been accelerated, and that of Saturn retarded. Halley, Euler, Lagrange, Laplace, and other eminent mathematicians, were led by their researches to the certain conclusion that these inequalities do not depend on the configuration of the orbits; and as Laplace proved that they are not occasioned by the action of comets, or bodies foreign to the system, he could only suppose them to belong to the class of periodic inequalities.

Observation had already shown that five times the mean motion of Saturn is so nearly equal to twice the mean motion of Jupiter, that the difference of these two quantities, or $5n' - 2n$, is an extremely small fraction, being about the 74th part of the mean motion of Jupiter. Laplace perceived that the square of this minute quantity is divisor to some of the perturbations in the longitude of Jupiter and Saturn, which led him to conjecture that the nearly commensurable ratio in the mean motions might be the cause of this anomaly in the theory of these two planets; a conjecture which computation amply confirmed, showing that a great inequality of $48'2''.207$ at its maximum exists in the theory of Saturn, which at the present time increases the mean motion of the planet, and accomplishes its changes in about 929 years; and that the mean motion of Jupiter is also affected by a corresponding and contrary inequality of nearly the same period, only amounting to $19'46''.62$ at its maximum, which diminishes the mean motion of Jupiter.

These two inequalities attained their maximum in the year 1560; from that period, the apparent mean motion of the two planets approached to their true motions, and became equal to them in 1790, which accounts for Halley finding the mean motion of Saturn slower, and that of Jupiter faster, by a comparison of ancient with modern observations, than modern observations alone allowed them to be: whilst on the other hand, modern observations indicated to Lambert[2] an acceleration in Saturn's motion, and a retardation in that of Jupiter; and the quantities of the inequalities found by these astronomers are nearly the same with those determined by Laplace.

Recorded observations of these mean motions at very remote periods enable us to ascertain the chronology of the nations in which science had made early advances. Thus the Indians determined the mean motions of Jupiter and Saturn, when the mean motion of Jupiter was at its maximum of acceleration, and that of Saturn at its greatest retardation; the two periods at which that was the case, were 3102 years before the Christian era, and 1491 years after it.

The formulae of the motions of Jupiter and Saturn determined by Laplace, agree with their oppositions, the error not amounting to $12''.96$, when it is to be recollected that only twenty years ago[3] the errors in the best tables exceeded $1296''$. These formulae also represent with great precision the observations of Flamstead,[4] of the Arabian astronomers,[5] and of Ptolemy,[6] leaving no grounds to doubt that Laplace has succeeded in solving this difficulty, by assigning the true cause of these inequalities, which had for so many ages baffled the acuteness of astronomers; so that anomalies which seemed at variance with the law of gravitation, do in fact furnish the strongest corroboration of the universal influence it exerts throughout the solar system. Such, says Laplace, has been the fate of that brilliant discovery of Newton, that every difficulty which has been raised against it, has formed a new subject of triumph, the sure characteristic of a law of nature.

The precision with which these two greatest planets of our system have obeyed the laws of mutual gravitation from the earliest periods at which we have records of their motions, proves the stability of the system, since Saturn has experienced no sensible action of foreign bodies from the time of Hipparchus,[7] although the sun's attraction on Saturn is about a hundred times less than that exerted on the earth.

Periodic Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the First Powers of the Disturbing Forces

572. If *i* be made equal to 5 in equation (169), the great inequality of Jupiter, including the secular variations of the elements of both orbits during its period of 929 years, is [8]

$$\begin{aligned} dv = dz = &+\frac{6m'n^2}{(5n'-2n)^2}\left\{\begin{array}{l}\left\{+aP'+\dfrac{2a\,.\,dP}{(5n'-2n)dt}-\&\text{c.}\right\}\cdot\sin(5n't-2nt+5\epsilon'-2\epsilon)\\ \left\{-aP+\dfrac{2a\,.\,dP'}{(5n'-2n)dt}+\&\text{c.}\right\}\cdot\cos(5n't-2nt+5\epsilon'-2\epsilon)\end{array}\right\}\\ &+\frac{2m'n}{5n'-2n}\left\{\begin{array}{l}a^2\cdot\dfrac{dP}{da}\cdot\cos(5n't-2nt+5\epsilon'-2\epsilon)\\ -a^2\cdot\dfrac{dP'}{da}\cdot\sin(5n't-2nt+5\epsilon'-2\epsilon)\end{array}\right\}\\ &-\frac{m'}{2}\cdot eK\,.\sin(5n't-2nt+5\epsilon'-2\epsilon-v+B)\\ &+\frac{5m'}{4}\cdot eK'\,.\sin(5n't-4nt+5\epsilon'-2\epsilon+v+B), \end{aligned} \qquad \textbf{(172)}$$

which must be applied as a correction to the mean motion of Jupiter.

573. Because of the equality and opposition of action and reaction, the great inequality in the mean motion of Saturn may be determined when that of Jupiter is known, and *vice versâ*; for by article 546,

$$\frac{dx^2+dy^2+dz^2}{dt^2}-2\frac{(S+m)}{r}=2\int \mathrm{d}R$$

may be assumed to belong to Jupiter, and

$$\frac{dx'^2+dy'^2+dz'^2}{dt^2}-2\frac{(S+m')}{r'}=2\int \mathrm{d}R'$$

to Saturn, $\mathrm{d}R$ and $\mathrm{d}R'$ relate to the co-ordinates of m and m'. Their sum, when the first equation is multiplied by m, and the second by m', is

$$2m\int \mathrm{d}R + 2m'\int \mathrm{d}R' = -2m\frac{(S+m)}{r} + m\frac{dx^2+dy^2+dz^2}{dt^2} - 2m'\frac{(S+m')}{r'} + m'\frac{dx'^2+dy'^2+dz'^2}{dt^2}.$$

The second member of this equation does not contain any term of the order of the squares of the disturbing masses having the divisor $5n't-2nt$, which can only arise from the integration of the sines or cosines of the angle $5n't-2nt$; because, when the elliptical values are substituted instead of x, y, z, the part [9]

$$-2m\frac{(S+m)}{r} + m\frac{dx^2+dy^2+dz^2}{dt^2},$$

will only contain the sines or cosines of the angle nt, and the remaining part of the second member is a function of $n't$ only; and as such terms as have the square of the divisor $5n't-2nt$ are alone under consideration, the second member may be omitted, then

$$m\int \mathrm{d}R + m'\int \mathrm{d}R' = 0. \qquad \textbf{(173)}$$

574. When $S+m=m$ is restored, which has hitherto been assumed equal to unity, the general expression for the periodic inequality in the mean motion of Jupiter is

$$dz = -3\iint \cdot \frac{andt\,.\,\mathrm{d}R}{S+m}.$$

The corresponding inequality in the mean motion of Saturn is

$$dz' = -3\iint \cdot \frac{a'n'dt\,.\,\mathrm{d}R'}{S+m'}.$$

From these two it is easy to find [10]

$$m(S+m)\,.\,a'n'\,.\,dz + m'(S+m')\,.\,an\,.\,dz' + 3m\,.\,a'n'\iint andt\,.\,\mathrm{d}R + 3m'\,.\,an\iint a'n'dt\,.\,\mathrm{d}R' = 0.$$

And in consequence of equation (173)[11]

$$m(S+m)\,.\,a'n'\,.\,dz + m'(S+m')\,.\,an\,.\,dz' = 0.$$

But

$$n = \frac{\sqrt{S+m}}{a^{\frac{3}{2}}} \quad n' = \frac{\sqrt{S+m'}}{a'^{\frac{3}{2}}};$$

and if the masses m and m' be omitted in $(S+m)$, $(S+m')$; in comparison of the mass of the sun taken as the unit, the preceding equation becomes

$$m\sqrt{a}\,.dz = -m'\sqrt{a'}\,.dz'\,.$$

Thus the periodic inequality in the mean motion of Jupiter is contrary to that in the mean motion of Saturn when n and n' have the same sines, which must always be the case, because both planets revolve about the sun in the same direction, so that one body is accelerated when the other is retarded, which corresponds with observation. These inequalities are in the ratio of $m\sqrt{a}$ to $m'\sqrt{a'}$; hence, if the inequality in the mean motion of Jupiter be known, that in the mean motion of Saturn will be found from

$$dz' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}dz\,. \qquad \textbf{(174)}$$

575. As the whole of the following analyses depends[12] on the angle $5n't - 2nt + 5\epsilon' - 2\epsilon$, it will be represented by l for the sake of abridgement. If i be made equal to 5 in equation (167), it becomes

$$R = m'P\,.\sin l + m'P'\,.\sin l\,.$$

From this, values of dR, $\frac{dR}{de}$, $\frac{dR}{dv}$, may be found; but equations (165) and (166), show that

$$\left(\frac{dP}{dv}\right) = e\left(\frac{dP'}{de}\right);\quad \left(\frac{dP'}{dv}\right) = -e\left(\frac{dP}{de}\right);$$

consequently, by the substitution of dR, $\frac{dR}{de}$, $\frac{dR}{dv}$ in equations (114), the periodic variations in the eccentricity, longitude of the perihelion, and semigreater axis of Jupiter's orbit, depending on the third powers of the eccentricities and inclinations, are easily found to be

$$de_{/} = +\frac{m'.an}{5n'-2n}\left\{\frac{dP}{de}\cdot\sin l + \frac{dP'}{de}\cdot\cos l\right\} \qquad \textbf{(175)}$$

$$edv_{/} = -\frac{m'.an}{5n'-2n}\left\{\frac{dP}{de}\cdot\cos l - \frac{dP'}{de}\cdot\sin l\right\}. \qquad \textbf{(176)}$$

576. The periodic inequalities in g and Π, the mutual inclination of the orbits of Jupiter and Saturn, and the longitude of the ascending node of the orbit of Saturn on that of Jupiter, are obtained from

$$R=\frac{m'}{4}\cdot Q_4e'g^2\,.\cos(l-2\Pi-v')+\frac{m'}{4}\cdot Q_5eg^2\,.\cos(l-2\Pi-v);$$

or

$$R=+\frac{m'}{4}\cdot g^2\cos2\Pi\left\{Q_4\,.\,e'\cos(l-v')+Q_5\,.\,e\cos(l-v)\right\}$$
$$+\frac{m'}{4}\cdot g^2\sin2\Pi\left\{Q_4\,.\,e'\sin(l-v')+Q_5\,.\,e\sin(l-v)\right\};$$

or to abridge

$$R=\frac{m'}{4}\cdot g^2\cos2\Pi\,.\,A+\frac{m'}{4}\cdot g^2\sin2\Pi\,.\,B.$$

But from article 444 it appears that

$$g^2\,.\cos2\Pi=(q'-q)^2-(p'-p)^2;\quad g^2\,.\sin2\Pi=2(q'-q)^2(p'-p)^2;$$

whence

$$R=\frac{m'}{4}\left\{(q'-q)^2-(p'-p)^2\right\}.\,A+\frac{m'}{4}\cdot2(q'-q)(p'-p)\,.B,$$

and

$$\frac{dR}{dp}=\frac{m'}{2}(p'-p)\,.\,A-\frac{m'}{2}(q'-q)\,.\,B,$$

or

$$\frac{dR}{dp}=\frac{m'}{2}.g\sin\Pi\,.A-\frac{m'}{2}\,g\cos\Pi\,.B;$$

restoring the values of A and B, and reducing the products of the sines and cosines,

$$\frac{dR}{dp}=-\frac{m'}{2}\cdot Q_4\,.\,e'g\,.\sin(l-v-\Pi)-\frac{m'}{2}\cdot Q_5\,.\,eg\cos(l-v-\Pi).\sin(l-v-\Pi),$$

But

$$\sin(l-v'-\Pi)=\sin(l+\Pi).\cos(v+2\Pi)-\cos(l+\Pi).\sin(v+2\Pi),$$

hence[13]

$$\frac{dR}{dp}=+\frac{m'}{2}\left\{Q_4\,.\,e'g\,.\sin(v'+2\Pi)+Q_5\,.\,eg\,.\sin(v+2\Pi)\right\}.\cos(l+\Pi)$$
$$+\frac{m'}{2}\left\{Q_4\,.\,e'g\,.\cos(v'+2\Pi)+Q_5\,.\,eg\,.\cos(v+2\Pi)\right\}.\sin(l+\Pi);$$

and from equations (165) and (166) it is clear that

$$\frac{dR}{dp} = m'\frac{dP}{dg}\cdot\cos\left(l+\Pi\right) - m'\frac{dP'}{dg}\cdot\sin\left(l+\Pi\right).$$

In the same manner it may be found that

$$\frac{dR}{dq} = -m'\cdot\frac{dP}{dg}\cdot\sin\left(l+\Pi\right) - m'\cdot\frac{dP'}{dg}\cdot\cos\left(l+\Pi\right);$$

with these values the two last of equations (114) become, when integrated,

$$d\,p = +\frac{m'.an}{5n'-2n}\cdot\left\{\frac{dP}{dg}\cdot\cos\left(l+\Pi\right) - \frac{dP'}{dg}\cdot\sin\left(l+\Pi\right)\right\};$$
$$d\,q = -\frac{m'.an}{5n'-2n}\cdot\left\{\frac{dP}{dg}\cdot\sin\left(l+\Pi\right) + \frac{dP'}{dg}\cdot\cos\left(l+\Pi\right)\right\}.$$

If s be the latitude of Jupiter, by article 436

$$s = q\sin v - p\cos v;$$

hence

$$d\,s = d\,q\,.\sin v - d\,p\,.\cos v,$$

and substituting for $d\,p,\ d\,q,$

$$d\,s = -\frac{m'.an}{5n'-2n}\left\{\frac{dP}{d\,l}\cdot\cos\left(l-v+\Pi\right) - \frac{dP'}{d\,l}\cdot\sin\left(l-v+\Pi\right)\right\} \qquad \textbf{(177)}$$

which is the only sensible inequality in the latitude of Jupiter in this approximation.

The latitude of Jupiter above the primitive orbit of Saturn is

$$s = -g\,\sin\left(v-\Pi\right)$$

whence

$$-d\,s = dg\,\sin\left(v-\Pi\right) - gd\Pi\cos\left(v-\Pi\right)$$

and a comparison of the two values of $d\,s,$ gives

$$dg' = \frac{m'.an}{5n'-2n}\left\{\frac{dP'}{dg}\cos\,l + \frac{dP}{dg}\sin\,l\right\}$$
$$gd\Pi' = -\frac{m'.an}{5n'-2n}\left\{\frac{dP}{dg}\cdot\cos\,l - \frac{dP'}{dg}\cdot\sin\,l\right\}.$$

These are the variations occasioned by the action of Saturn in the mutual inclination of the two orbits, and in the ascending node of their common intersection; but Jupiter produces a corresponding effect in these two quantities, and if it be expressed by dg'', $gd\Pi''$; then the whole variations will be

$$dg = dl' + dg'', \quad d\Pi = d\Pi' + d\Pi'';$$

but by article

$$dg'' = \frac{m.a'n'}{m'.an} \cdot dg'; \quad d\Pi'' = \frac{m.a'n'}{m'.an} \cdot d\Pi';$$

or, substituting for n and n', the whole variations in the two elements in question are

$$\begin{aligned} dg &= +\frac{m'.an}{5n'-2n} \cdot \frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ \frac{dP'}{dg}\cos l + \frac{dP}{dg}\sin l \right\} \\ gd\Pi &= +\frac{m'.an}{5n'-2n} \cdot \frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ \frac{dP'}{dg}\sin l - \frac{dP}{dg}\cos l \right\}. \end{aligned} \qquad \textbf{(178)}$$

577. The corresponding periodic inequalities in the latitude and elements of the orbit of Saturn are[14]

$$\begin{aligned} ds &= -\frac{2a'n'.m}{(5n'-2n)m'} \left\{ \begin{aligned} &+\frac{dP'}{dg}\sin\{4n't - 2nt + 4\epsilon' - 2\epsilon - v + \Pi\} \\ &-\frac{dP}{dg}\cos\{4n't - 2nt + 4\epsilon' - 2\epsilon - v + \Pi\} \end{aligned} \right\}, \\ dz' &= -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot dz, \\ de' &= +\frac{m.a'n'}{5n'-2n} \left\{ \frac{dP'}{de'}\cos l + \frac{dP}{de'}\sin l \right\}, \\ e'dv' &= -\frac{m.a'n'}{5n'-2n} \left\{ \frac{dP}{de'}\cos l - \frac{dP'}{de'}\sin l \right\}. \end{aligned} \qquad \textbf{(179)}$$

It is evident that the variations in the mean motions are by much the greatest, on account on account of the divisor $(5n'-2n)^2$.

Periodic Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the Squares of the Disturbing Forces

578. The equations in the preceding articles, which determine the periodic inequalities in the elements of the orbits of Jupiter and Saturn, are functions of the sines and cosines of their mean motions; and when the mean motions are corrected by the application of their great inequalities, the equations in question give secular as well as periodic inequalities in the elements of both orbits; depending on the squares and products of the disturbing masses.

The great inequalities may be put under a convenient form for this analysis, if the value of *R*, in article 563, be expressed by

$$R = m'.\Sigma.Q.\cos\{5n't - 2nt + 5\epsilon' - 2\epsilon - b\},$$

where b is a function of the longitudes of the perihelia and node of the common intersection of the two orbits. The substitution of this in

$$dz = -3\iint.andt.dR,$$

gives

$$dz = -6m'\iint.an^2dt^2.\Sigma Q.\sin(5n't - 2nt + 5\epsilon' - 2\epsilon - b). \qquad \textbf{(180)}$$

Since dz and dz' represent the great inequalities of Jupiter and Saturn, their corrected mean motions are $nt + dz$, and $n't + dz'$; and, by the substitution of these in the preceding equation, it becomes

$$(dz) = -6m'\iint.an^2dt^2.\Sigma Q.\sin\{5n't - 2nt + 5\epsilon' - 2\epsilon - b + 5dz' - 2dz\} \qquad \textbf{(181)}$$

(dz) being the great inequality of Jupiter when the mean motions are corrected. In order to abridge, let

$$5n't - 2nt + 5\epsilon' - 2\epsilon = l,$$

then

$$\sin(l - b + 5dz' - 2dz) = \sin(l - b)\cos(5dz' - 2dz) + \cos(l - b)\sin(5dz' - 2dz).$$

But $5dz' - 2dz$ is so small, that it may be taken for its sine, and unity for its cosine; and as quantities of the order of the square of the disturbing forces are alone to be retained, $\sin(l - b)$ may be omitted; hence

$$\sin(l - b + 5dz' - 2dz) = \{5dz' - 2dz\}\cos(l - b);$$

or, as

$$dz' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot dz$$

therefore

$$\sin(l - b + 5dz' - 2dz) = -\left\{\frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}}\right\}\cdot dz\ .\cos(l - b)$$

but the integral of equation (180) is

$$dz = \frac{6m'.an^2.\Sigma.Q}{(5n'-2n)^2}\cdot\sin(l-b),$$

consequently [15]

$$\sin(l-b+5dz'-2dz) = -\frac{(3m'.an^2.\Sigma.Q)}{(5n'-2n)^2}\cdot\left\{\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\right\}\cdot\sin(2l-2b).$$

When this quantity is substituted in equation (181), instead of the sine, its integral

$$(dz) = -\frac{(3m'.an^2.\Sigma.Q)^2}{2(5n'-2n)^4}\left\{\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\right\}\sin 2(5n't-2nt+5\epsilon'-2\epsilon-b) \qquad \textbf{(182)}$$

is the variation in the mean motion of Jupiter, and on account of the relation in article 574, the corresponding inequality in the mean motion of Saturn is

$$(dz') = \frac{(3m.an^2.\Sigma.Q)^2}{2(5n'-2n)^4}\left\{\frac{5m'\sqrt{a'}+2m\sqrt{a}}{m'\sqrt{a'}}\right\}\frac{m\sqrt{a}}{m'\sqrt{a'}}\sin 2(5n't-2nt+5\epsilon'-2\epsilon). \qquad \textbf{(183)}$$

These inequalities have a sensible effect, on account of the minute divisor $(5n'-2n)^4$.

579. The great inequalities in the mean motions also occasion variations in the eccentricities and longitudes of the perihelia, depending on the squares of the disturbing forces.

The principal term of the great inequality is sufficient for this purpose; and if the secular variations in the elements of the orbits during the period of the inequalities be omitted, the first term of the great inequality in the mean motion of Jupiter (172), when l is put for $5n't-2nt+5\epsilon'-2\epsilon$, is,

$$-\frac{6m'.an^2}{(5n'-2n)^2}\{P\cos l - P'\sin l\}.$$

The corresponding inequality in the mean motion of Saturn is

$$+\frac{6m'.an^2}{(5n'-2n)^2}\cdot\frac{m\sqrt{a}}{m'\sqrt{a'}}\{P\cos l - P'\sin l\}.$$

If these be applied as corrections to nt and $n't$, in the differential of equation (175), or

$$dde = +m'.andt\cdot\left\{\frac{dP}{de}\cdot\cos l - \frac{dP'}{de}\cdot\sin l\right\},$$

it will be found, by the same analysis that was employed in the last article, that

$$\begin{aligned} d\,.de &= +m'.andt\left\{\frac{dP}{de}\cdot\cos l - \frac{dP'}{de}\cdot\sin l\right\} \\ &-m'.andt\frac{dP}{de}\left\{\frac{6m'.an^2}{(5n'-2n)^2}\frac{5m'\sqrt{a'}+2m\sqrt{a}}{m'\sqrt{a'}}\right\}\left\{P.\cos l\sin l - P'\sin^2 l\right\} \\ &-m'.andt\frac{dP'}{de}\left\{\frac{6m'.an^2}{(5n'-2n)^2}\frac{5m\sqrt{a'}+2m\sqrt{a}}{m'\sqrt{a'}}\right\}\left\{P.\cos^2 l - P'\cos l\sin l\right\}. \end{aligned} \qquad \textbf{(184)}$$

But

$$P\cos l\sin l - P'\sin^2 l = \tfrac{1}{2}P\sin 2l + \tfrac{1}{2}P'\cos 2l - \tfrac{1}{2}P'$$
$$P\cos^2 l - P'\cos l\sin l = \tfrac{1}{2}P\cos 2l - \tfrac{1}{2}P'\sin 2l + \tfrac{1}{2}P;$$

and, as terms depending on the first powers of the masses are to be rejected, the periodic part of the preceding equation is

$$\begin{aligned} de_2 = &-\frac{3m'^2.a^2n^3}{2(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P'\cdot\frac{dP}{de}+P\cdot\frac{dP'}{de}\right\}\times \\ &\sin 2(5n't-2nt+5\epsilon'-2\epsilon) \\ &-\frac{3m'^2.a^2n^3}{2(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P'\cdot\frac{dP'}{de}-P\cdot\frac{dP}{de}\right\}\times \\ &\cos 2(5n't-2nt+5\epsilon'-2\epsilon). \end{aligned} \qquad \textbf{(185)}$$

By the same process it may be found that the periodic variations of *nt*, and $n't$, produce the periodic variation

$$\begin{aligned} dv_2 = &+\frac{3m'^2.a^2n^3}{2e(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP}{de}-P'\cdot\frac{dP'}{de}\right\}\times \\ &\sin 2(5n't-2nt+5\epsilon'-2\epsilon) \\ &+\frac{3m'^2.a^2n^3}{2e(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P'\cdot\frac{dP}{de}-P\cdot\frac{dP'}{de}\right\}\times \\ &\cos 2(5n't-2nt+5\epsilon'-2\epsilon), \end{aligned} \qquad \textbf{(186)}$$

in the longitude of the perihelion of Jupiter. These are the only sensible periodic inequalities in the elements of Jupiter's orbit of this order. Corresponding variations obtain in those of the orbit of Saturn.

Secular Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the Squares of the Disturbing Forces

580. The secular variations in the elements of the orbits of Jupiter and Saturn depending on the first powers of the disturbing forces, are determined by the formulae (130), in common with the other planets; but to these must be added their variations depending on the squares of the masses, quantities only sensible in the motions of Jupiter and Saturn.
The secular part of equation (184), arising from the corrected values of $nt,\ n't,$ is

$$(de) = -\frac{3m'^2 \,.\, a^2n^3}{(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP'}{de}-P'\cdot\frac{dP}{de}\right\}. \qquad \textbf{(187)}$$

and the corresponding variation in the longitude of the perihelion of Jupiter's orbit, depending on the squares of the disturbing forces, is

$$(dv) = \frac{3m'^2 \,.\, a^2n^3}{e(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP}{de}+P'\cdot\frac{dP'}{de}\right\}. \qquad \textbf{(188)}$$

The corresponding inequalities for Saturn are,

$$(de') = -\frac{3m^2 \,.\, a^2n^3}{a'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\frac{dP'}{de'}-P'\cdot\frac{dP}{de'}\right\}$$
$$(dv') = \frac{3m^2 \,.\, a^3n^3}{a'e'(5n'-2n)}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP}{de'}+P'\cdot\frac{dP'}{de'}\right\}. \qquad \textbf{(189)}$$

581. Thus the periodic inequalities in the mean motions cause both periodic and secular variations in the elements of the two orbits of the order of the squares of the disturbing forces; but the periodic variations in the other elements have the same effect; for, making

$$5n't - 2nt + 5\epsilon' - 2\epsilon = l,$$

the differential of equation (175) is

$$d\epsilon = +m'\cdot andt\left\{\frac{dP}{de}\cos l - \frac{dP'}{de}\sin l\right\};$$

and when all the elements are variable except the mean motion, the effects of which have already been determined,

$$d\,.\,de = +m'\,.\,andt\,.\begin{Bmatrix} -de\left(\frac{d^2P'}{de^2}\right)\cdot\sin\,l - \left(\frac{d^2P}{de^2}\right)\cdot\cos\,l \\ -dv\left(\frac{d^2P'}{dedv}\right)\cdot\sin\,l - \left(\frac{d^2P}{dedv}\right)\cdot\cos\,l \\ -de'\left(\frac{d^2P'}{dede'}\right)\cdot\sin\,l - \left(\frac{d^2P}{dede'}\right)\cdot\cos\,l \\ -dv'\left(\frac{d^2P'}{dedv'}\right)\cdot\sin\,l - \left(\frac{d^2P}{dedv'}\right)\cdot\cos\,l \\ -dg\left(\frac{d^2P'}{dedg}\right)\cdot\sin\,l - \left(\frac{d^2P}{ded\,l}\right)\cdot\cos\,l \\ -d\Pi\left(\frac{d^2P'}{ded\Pi}\right)\cdot\sin\,l - \left(\frac{d^2P}{ded\Pi}\right)\cdot\cos\,l \end{Bmatrix}$$

If the values of dv, de, de', dv', dg, and $d\Pi$, from article 575, and those that follow, be substituted, observing at the same time, that equations (165) and (166) give

$$\begin{aligned} &\frac{d^2P}{de\,.\,dv} = e\cdot\frac{d^2P'}{de^2}; \qquad \frac{d^2P'}{de\,.\,dv} = -e\cdot\frac{d^2P'}{de^2}; \\ &\frac{d^2P}{de\,.\,dv'} = e'\cdot\frac{d^2P'}{de\,.\,de'}; \qquad \frac{d^2P'}{de\,.\,dv'} = -e'\cdot\frac{d^2P'}{de\,.\,de'}; \\ &\frac{d^2P}{de\,.\,d\Pi} = g\cdot\frac{d^2P'}{de\,.\,dg}; \qquad \frac{d^2P'}{de\,.\,d\Pi} = -g\cdot\frac{d^2P}{de\,.\,dg}; \end{aligned} \qquad \textbf{(190)}$$

It will be found, when the periodic terms are omitted, and equation (187) added, that the whole secular variation in the eccentricity of Jupiter's orbit, depending on the squares of the disturbing forces, is

$$\begin{aligned} (de) = &-\frac{3m'^2\,.\,a^2n^3}{(5n'-2n)^2}\cdot t\cdot\frac{2m'\sqrt{a'}+5m\sqrt{a}}{m'\sqrt{a'}}\cdot\left\{P\cdot\left(\frac{dP'}{de}\right) - P'\cdot\left(\frac{dP}{de}\right)\right\} \\ &+\frac{m'^2\,.a^2\,.n^2}{5n'-2n}\cdot t\cdot\begin{Bmatrix} +\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P}{de^2}\right) - \left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P'}{de^2}\right) \\ +\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{dedg}\right) - \left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{dedg}\right) \end{Bmatrix} \end{aligned}$$

$$+\frac{mm'.aa'.nn'}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de.de'}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de.dg}\right)\end{array}\right\}. \qquad \textbf{(191)}$$

By the same process it may be found, that where the periodic terms which are quite insensible are omitted, the secular variation in the longitude of the perihelion of Jupiter's orbit, depending on the squares of the disturbing force, including the equation (188), is[16]

$$(dv)=+\frac{3m'^2.a^2n^3}{e(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\left(\frac{dP}{de}\right)+P'\cdot\left(\frac{dP'}{de}\right)\right\}$$

$$+\frac{m'^2.a^2.n^2}{e(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P}{de^2}\right)+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de^2}\right)\\+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{dedg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P'}{dedg}\right)\end{array}\right\} \qquad \textbf{(192)}$$

$$+\frac{mm'.aa'.nn'}{e(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P}{de.de'}\right)+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)\end{array}\right\}.$$

582. The corresponding variations for Saturn, including equations (189), are,

$$(de')=-\frac{3m^2.a^2n^3}{a'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\left(\frac{dP'}{de'}\right)-P'\cdot\left(\frac{dP}{de'}\right)\right\}$$

$$+\frac{m^2.a'^2.n'^2}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de'^2}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de'^2}\right)\\+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de'dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'dg}\right)\end{array}\right\} \qquad \textbf{(193)}$$

$$+\frac{mm'.aa'.nn'}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P}{de.de'}\right)-\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\-\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de'.dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'.dg}\right)\end{array}\right\};$$

$$(dv') = +\frac{3m^2 \,.\, a^3 n^3}{a'e'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\left(\frac{dP}{de'}\right)+P'\cdot\left(\frac{dP'}{de'}\right)\right\}$$
$$+\frac{m^2 \,.\, a'^2 \,.\, n'^2}{e'(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P}{de'^2}\right)+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P'}{de'^2}\right)\\ +\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de'dg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P'}{de'dg}\right)\end{array}\right\}\qquad\textbf{(194)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{e'(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P}{de\,.\,de'}\right)+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de\,.\,de'}\right)\\ +\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de'\,.\,dg}\right)+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'\,.\,dg}\right)\end{array}\right\}.$$

583. Secular variations, depending on the squares of the disturbing forces, arise from the same cause in the mutual inclination of the orbits, and in the longitude of the ascending mode of the orbit of Saturn on that of Jupiter. These are obtained from equations (178), considering the elements to be variable; then the substitution of their periodic variations will give, in consequence of

$$\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{dg^2}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{dg^2}\right)=0.$$

$$(dg) = -\frac{3m'^2 \,.\, a^2 n^3}{(5n'-2n)^2}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\left\{P\cdot\left(\frac{dP'}{dg}\right)-P'\cdot\left(\frac{dP}{dg}\right)\right\}$$
$$+\frac{m'^2 \,.\, a^2 \,.\, n^2}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P}{de\,.\,dg}\right)-\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P'}{de\,.\,dg}\right)\right\}\qquad\textbf{(195)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de'\,.\,dg}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de'\,.\,dg}\right)\right\};$$

$$(d\Pi) = +\frac{3m'^2 \,.\, a^2 n^3}{g\,(5n'-2n)^2}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\left\{P\left(\frac{dP}{dg}\right)+P'\left(\frac{dP'}{dg}\right)\right\}$$
$$+\frac{m'^2 \,.\, a^2 \,.\, n^2}{g\,(5n'-2n)}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\left\{\left(\frac{dP}{de}\right)\left(\frac{d^2P}{de\,.\,dg}\right)+\left(\frac{dP'}{de}\right)\left(\frac{d^2P'}{de\,.\,dg}\right)\right\}\qquad\textbf{(196)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\left(\frac{d^2P}{de'\,.\,dg}\right)-\left(\frac{dP}{de'}\right)\left(\frac{d^2P'}{de'\,.\,dg}\right)\\ +\left(\frac{dP}{dg}\right)\left(\frac{d^2P}{dg^2}\right)+\left(\frac{dP'}{dg}\right)\left(\frac{d^2P'}{dg^2}\right)\end{array}\right\};$$

584. These are the variations with regard to the plane of Jupiter's orbit at a given time, but the variations in the position of the orbits of Jupiter and Saturn with regard to the ecliptic may easily be found, for f, f', being the inclinations of the orbits of m and m' on the fixed ecliptic at the epoch, and q, q', the longitudes of the ascending nodes estimated on that plane, by article 444,

$$p' - p = g \sin \Pi; \qquad q' - q = g \cos \Pi;$$

or

$$f' \sin q' - f \sin q = g \sin \Pi,$$
$$f' \cos q' - f \cos q = g \cos \Pi.$$

and on account of the action and reaction of Jupiter and Saturn,

$$d\left(f'.\sin q'\right) = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot d\left(f \sin q\right),$$
$$d\left(f'.\cos q'\right) = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot d\left(f \cos q\right).$$

And from these four equations, it will readily be found, that

$$\begin{aligned}
(df) &= -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg.\cos\left(\Pi - q\right) - gd\Pi \sin\left(\Pi - q\right)\right\}\\
(fdq) &= -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg.\sin\left(\Pi - q\right) + gd\Pi \cos\left(\Pi - q\right)\right\}\\
(df') &= \frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg.\cos\left(\Pi - q'\right) - gd\Pi \sin\left(\Pi - q\right)\right\}\\
(f'dq') &= \frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg.\cos\left(\Pi - q'\right) + gd\Pi \cos\left(\Pi - q'\right)\right\}.
\end{aligned} \qquad \textbf{(197)}$$

Thus when dg and $gd\Pi$ are computed, the variations in the inclinations and longitude of the nodes when referred to the fixed plane of the ecliptic may be found.

585. The periodic variations in the eccentricities, inclinations, longitudes of the perihelia, and nodes, do not affect the mean motion with any sensible inequalities depending on the squares and product of the masses; for if the variation of

$$dz = +\frac{6m'.an^2}{\left(5n' - 2n\right)^2}\left\{P'.\sin l - P.\cos l\right\}$$

be taken, considering all the elements as variable, the substitution of their periodic variations will make the whole vanish in consequence of the relations between the partial differences.

586. The longitude of the epoch is not affected by any variations of this order that are sensible in the planets, but they are of much importance in the theories of the moon and Jupiter's satellites.

587. The variations in the elements depending on the squares of the disturbing forces, are insensible in the theories of all the planets, except those of Jupiter and Saturn; they are only perceptible in the motions of these two planets, on account of the nearly commensurable ratio in their mean motions introducing the minute divisor $5n' - 2n$; therefore, if

$$(d\bar{e}),\ (d\bar{v}),\ (d\bar{g}),\ (d\bar{\Pi}),\ (d\bar{f}),\ (d\bar{q}),$$

be the secular variations in the elements depending on the second powers of the disturbing forces, and computed for the epoch from the equations in articles 580, and the two following, the equations (130) become, with regard to Jupiter and Saturn only,[17]

$$\begin{aligned}
e &= \bar{e} + \left\{\frac{d\bar{e}}{dt} + (d\bar{e})\right\} t + \&\text{c.} \\
v &= \bar{v} + \left\{\frac{d\bar{v}}{dt} + (d\bar{v})\right\} t + \&\text{c.} \\
g &= \bar{g} + \left\{\frac{d\bar{g}}{dt} + (d\bar{g})\right\} t + \&\text{c.} \\
\Pi &= \bar{\Pi} + \left\{\frac{d\bar{\Pi}}{dt} + (d\bar{\Pi})\right\} t + \&\text{c.} \\
f &= \bar{f} + \left\{\frac{d\bar{f}}{dt} + (d\bar{f})\right\} t + \&\text{c.} \\
q &= \bar{q} + \left\{\frac{d\bar{q}}{dt} + (d\bar{q})\right\} t + \&\text{c.}
\end{aligned} \qquad \textbf{(198)}$$

Whence the elements of the orbits of these two planets may be determined with greater accuracy for 1,000 or 1,200 years before and after the time assumed as the epoch.

Periodic Perturbations in Jupiter's Longitude depending on the Squares of the disturbing Forces

588. Where e^2 is omitted, equation (97) becomes

$$v = 2e\sin\left(nt + \epsilon - v\right).$$

The eccentricity and longitude of the perihelion, when corrected for their periodic inequalities (175), and (185) (176) and (186), become,

$$e + de_1 + de_2 \text{ and } v + dv_1 + dv_2,$$

and the longitude of the epoch when corrected by its periodic variation, is $\in + d\in_1$; by the substitution of these v becomes

$$dv = (2e + 2de_1 + 2de_2)\sin\{nt + \in - v + d\in_1 - dv_1 - dv_2\}:$$

when the quantities that do not contain the squares of the disturbing forces are rejected, the development of this expression is

$$dv = \{2de_2 + 2edv_1 . d\in_1 - edv^2\}\sin(nt + \in - v) - \{2dv_1 + 2ede_1 . dv_1 - ede . d\in\}\cos(nt + \in - v);$$

when the values of the periodic variations are substituted, the result will be the inequality[18]

$$\begin{aligned} dv = &-\frac{3m'^2 . a^2n^3}{(5n' - 2n)^3} \cdot \frac{5m\sqrt{a} + 4m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{P \cdot \frac{dP'}{de} + P' \cdot \frac{dP}{de}\right\} \times \\ &\cos\{5nt - 10n't + 5\in - 10\in' - v\} \\ &-\frac{3m'^2 . a^2n^3}{(5n' - 2n)^3} \cdot \frac{5m\sqrt{a} + 4m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{P' \cdot \frac{dP'}{de} - P \cdot \frac{dP}{de}\right\} \times \\ &\sin\{5nt - 10n't + 5\in - 10\in' - v\}. \end{aligned} \qquad \textbf{(199)}$$

The corresponding inequality for Saturn is found from

$$v' = 2e'\sin(n't + \in - v').$$

589. The radii vectores and true longitudes of m and m' in their elliptical orbits have been represented by r, r', v, v', but as

$$dr,\ dr',\ dv,\ dv'$$

are the periodic perturbations of these quantities, these two co-ordinates of m and m' in their troubled orbits, are

$$r + dr,\ r' + dr',\ v + dv,\ v' + dv'.$$

When these quantities are substituted in

$$R = \frac{m'\left(rr'\cos\left(v'-v\right)\right)+zz'}{\left(r'^2+z'^2\right)^{\frac{3}{2}}} - \frac{m'}{\sqrt[3]{r^2-2rr'\cos\left(v'-v\right)+r'^2}},$$

R becomes a function of the squares and products of the masses, it consequently produces terms of that order in the mean motion

$$z = -3\iint . andt . \mathrm{d}R$$

having the factor $\left(5n'-2n\right)^2$; they therefore form a part of the great inequalities in the mean motions of Jupiter and Saturn. A mistake has been observed in Laplace's determination of those inequalities,[19] which has been, and still is, a subject of controversy between three of the greatest mathematicians of the present age, MM. Plana,[20] Poisson,[21] and Pontécoulant,[22] to whose very learned papers the reader is referred for a full investigation of this difficult subject.

590. The numerical values of the perturbations of Jupiter in longitude are computed from equations (159), (164), (172), (182), and (199), together with some terms depending on the fifth powers of the eccentricities and inclinations which may be determined by the same process as in the other approximations; his perturbations in latitude are computed from equations (160) and (177), and those in his radius vector from (158) and (163).

591. Hitherto the mass of the planet has been omitted when compared with that of the sun taken as the unit; so that half the greater axes has been determined by the equation $a^3 = \frac{1}{n^2}$, whereas its real value is found from

$$\frac{1+m}{a^3} = n^2, \text{ or } a = n^{-\frac{3}{2}}\left(1+\tfrac{1}{3}m\right);$$

the semigreater axes of the orbits of Jupiter and Saturn ought therefore to be augmented by $\frac{1}{3}ma$, $\frac{1}{3}m'a'$, quantities that are only sensible in these two planets.

Notes

[1] See note 55, *Preliminary Dissertation.*
[2] Lambert, Johann Heinrich, 1728-1777, self-educated mathematician, bom in Mülhausen, Germany. One of the first to appreciate the nature of the Milky Way. He became editor of the astronomical almanac *Astronomisches Jahrbuch oder Ephemeriden* in 1774. In his *Theorie der Parallellinien* (1766) he developed several pioneering but inconclusive theorems in non-Euclidean geometry. In 1768 he demonstrated the irrational nature of the number pi (p). Lambert also developed the first methods for the measurement of light intensity (1760). A unit of light intensity is named after him.
[3] That is around 1810.

[4] Flamstead or Flamsteed, John, 1646-1719, the first Astronomer Royal (1675-1719). He was born in Denby, England. Flamsteed instituted reliable observations at Greenwich in 1676 (see note 22, *Preliminary Dissertation*). Isaac Newton (see note 1, *Preliminary Dissertation*) used Flamsteed's data to verify his gravitational theory. His star catalog *Historia Coelestis Britannica* (1725) lists over 3000 stars with great precision.

[5] *Arabian astronomers.* Somerville is likely referring to the Muslin astronomers al-Farghani (c. 860), al-Battani (868-929), and Thabit ibn Qurrah (c. 836-901). Al-Farghani's *Kitab fi al-Harakat al-Samawiya wajawami Ilm al-Nujum* (Elements of Astronomy) was translated into Latin in the 12th century and exerted great influence upon European astronomy. Al-Battani made a remarkably accurate determination of the solar year as being 365 days, 5 hours, 46 minutes and 24 seconds, which is very close to modern values. He found that the longitude of the sun's apogee had increased by 16° 47' since Ptolemy. This implied the important discovery of the motion of the solar apsides and of a slow variation in the equation of time. He also measured the obliquity of the ecliptic, the length of the seasons and the true and mean orbit of the sun. His measurements of lunar and solar eclipses were used in 1749 to determine the secular acceleration of the moon. Thabit ibn Qurrah was one of the early reformers of Ptolemaic views. He analysed several problems related to the movements of sun and moon and wrote treatises on sun-dials.

[6] See note 15, *Preliminary Dissertation.*

[7] See note 32, *Preliminary Dissertation.*

[8] In the 1st edition term 4 *reads* $+\frac{5m_{/}}{2}\cdot eK\,.\sin\left(5n't-2nt+5\epsilon'-2\epsilon+v+b\right)$ and B in term 3 *reads* b In addition a final term $+m'.He.\sin\left(5n't-2nt+5\epsilon'-2\epsilon+v+b\right)$ is deleted (published errata). In the 1st edition the first term is expressed as two terms with identical coefficients but opposite signs.

[9] The 1st term *reads* $-2m\frac{(s+m)}{r}$ in the 1st edition.

[10] In the 1st edition dz' in the second term has no accent (published erratum).

[11] As above, in the 1st edition dz' in the second term has no accent (published erratum).

[12] This *reads* "depend" in the 1st edition (published erratum).

[13] The coefficient of second term *reads* $+\frac{m'}{2}$ in the 1st edition (published erratum).

[14] The elements $\frac{dP'}{dl}$ and $\frac{dP}{dl}$ in the first equation are reversed in the 1st edition (published erratum).

[15] $\left(3m'.an^2.\sum.Q\right)$ *reads* as its square in the 1st edition (published erratum).

[16] The original pagination in this area of the 1st edition text is out of sequence. The numbering *reads* 337, 338, 337 (repeated), 338 (repeated), 339.

[17] The 4th equation *reads* $p=\bar{p}+\left\{\frac{d\bar{\Pi}}{dt}+(d\bar{\Pi})\right\}t+\text{\&c.}$ in the 1st edition (published erratum).

[18] The arguments in the 3rd and 4th lines *read* $5n't-10nt+5\epsilon'-10\epsilon-v$ in the 1st edition (published erratum).

[19] See note 4, *Introduction.*

[20] Plana, Giovanni Antonio Amedeo, barone, 1781-1864, *Integration des formules propres a determiner les equations seculaires des elements des planetes et des cometes : produites par la resistance d'un milieu tres-rare*, Genes : Imprimerie Carniglia, 1825.

[21] See note 1, *Book I, Chapter VI.*

[22] See note 3, *Book II, Chapter IV.*