BOOK II

CHAPTER VII

PERIODIC VARIATIONS IN THE ELEMENTS OF THE PLANETARY ORBITS

Variations depending on the first Powers of the Eccentricities and Inclinations

529. THE differential d*R* relates to the arc *nt* alone, consequently the differential equation $da = 2a^2 \,.\, \mathrm{d}R$ in article 439 becomes

$$\begin{aligned} da = &+m'a^2 \,.\, in \,.\, \sum A_i \sin i\left(n't - nt + \epsilon' - \epsilon\right) \\ &+m'a^2 en\left(i-1\right).\, M_0 \sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v\right\} \\ &+m'a^2 e' n\left(i-1\right).\, M_1 \sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v'\right\}. \end{aligned}$$

The integral of this equation is the periodic variation in the mean distance, and if represented by $d\,a$, then

$$\begin{aligned} d\,a = &-m'a^2 \frac{n}{n'-n} \cdot \sum A_i \cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ &-m'a^2 e \frac{\left(i-1\right)n}{i\left(n'-n\right)+n} M_0 \cos\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v\right\} \\ &-m'a^2 e' \frac{\left(i-1\right)n}{i\left(n'-n\right)+n} M_1 \cos\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v'\right\}. \end{aligned}$$

In a similar manner it may be found that the periodic variation in the mean motion $d\zeta = -3\int a n d t \mathrm{d}R$ is,

$$\begin{aligned} d\zeta = &\tfrac{3}{2} \cdot m'a \cdot \frac{n^2}{i\left(n'-n\right)^2} \cdot A_i \sin i\left(n't - nt + \epsilon' - \epsilon\right) \\ &+\tfrac{3}{2} \cdot m'a e \cdot \frac{\left(i-1\right)n^2}{\left\{i\left(n'-n\right)+n\right\}^2} \cdot M_0 \sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v\right\} \\ &+\tfrac{3}{2} \cdot m'a e' \cdot \frac{\left(i-1\right)n^2}{\left\{i\left(n'-n\right)+n\right\}^2} \cdot M_1 \sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - v'\right\}. \end{aligned}$$

From the other differential equations in article 439 it may also be found that the periodic variation in the eccentricity is[1]

$$
\begin{aligned}
de = {} & \tfrac{1}{2} m'a \frac{n}{i(n'-n)+n} M_0 \cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& + \tfrac{1}{4} m'ae \frac{n}{n'-n} A_i \cos i(n't - nt + \epsilon' - \epsilon) \\
& + m'ae' \frac{n}{i(n'-n)+2n} N_0 \cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\varpi\} \\
& + \tfrac{1}{2} m'a . e' \frac{n}{i(n'-n)+2n} N_1 \cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - \varpi - \varpi'\} \\
& - \tfrac{1}{2} m'a . e' \frac{n}{i(n'-n)} N_4 \cos\{i(n't - nt + \epsilon' - \epsilon) + \varpi - \varpi'\} \\
& + \tfrac{1}{2} m'a . e' \frac{n}{i(n'-n)} N_5 \cos\{i(n't - nt + \epsilon' - \epsilon) - \varpi + \varpi'\}.
\end{aligned}
$$

The variation of the epoch

$$
\begin{aligned}
d\epsilon = {} & -ma' \frac{n}{i(n'-n)} a \left(\frac{dA_i}{da}\right) \sin i(n't - nt + \epsilon' - \epsilon) \\
& + \tfrac{1}{4} m'ae \frac{n}{i(n'-n)+n} M_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& - m'a^2 e \frac{n}{i(n'-n)+n} \cdot \frac{dM_0}{da} \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& - m'a^2 e' \frac{n}{i(n'-n)+n} \cdot \frac{dM_1}{da} \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}.
\end{aligned}
$$

The variation in the longitude of the perihelion

$$
\begin{aligned}
ed\varpi = {} & \tfrac{1}{2} m'a \frac{n}{i(n'-n)+n} M_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& + m'ae \frac{n}{i(n'-n)+2n} N_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\varpi\} \\
& + mae \cdot \frac{n}{i(n'-n)} N_3 \sin i(n't - nt + \epsilon' - \epsilon) \\
& + \tfrac{1}{2} m'ae' \frac{n}{i(n'-n)+2n} N_1 \sin\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - \varpi - \varpi'\}
\end{aligned}
$$

$$+\tfrac{1}{2}m'ae'\frac{n}{i(n'-n)}N_4\sin\{i(n't-nt+\epsilon'-\epsilon)+\nu-\nu'\}$$
$$+\tfrac{1}{2}m'ae'\frac{n}{i(n'-n)}N_5\sin\{i(n't-nt+\epsilon'-\epsilon)-\nu+\nu'\}.$$

When e^2, eg, $e'g$, are omitted, the differentials of p and q in article 437 become

$$dp=a^2ndt\sin(nt+\epsilon)\frac{\mathrm{d}R}{dz}$$
$$dq=a^2ndt\cos(nt+\epsilon)\frac{\mathrm{d}R}{dz}.$$

When the orbit of m at the epoch is assumed to be the fixed plane,

$$z=0,\text{ and } z'=a'g\sin(n't+\epsilon'-\Pi).$$

the products of the inclination by the eccentricities being omitted.

Now although z be zero, its differential is not, therefore $\frac{\mathrm{d}R}{dz}$ must be determined from

$$R=\frac{m'zz'}{a'^3}+\frac{m'(z'-z)^2}{4}\Sigma B_i\cos i(n't-nt+\epsilon'-\epsilon);$$

whence

$$\frac{\mathrm{d}R}{dz}=\frac{m'z'}{a'^3}-\frac{m'z'}{2}\Sigma B_i\cos i(n't-nt+\epsilon'-\epsilon),$$

and

$$\frac{\mathrm{d}R}{dz}=\frac{-m'}{a'^2}g\sin\left\{(n't+\epsilon'-\Pi)+\frac{m'}{2}a'\Sigma B_{(i-1)}g\sin\{i(n't-nt+\epsilon'-\epsilon)+nt+\epsilon-\Pi\}\right\}$$

where i may be any whole number, positive or negative, except zero. When this quantity is substituted in dp, dq, their integrals are[2]

$$d\,p=-\frac{m'}{2}\cdot\frac{a^2n}{a'^2}g\left\{\frac{1}{n'-n}\sin(n't-nt+\epsilon'-\epsilon-\Pi)-\frac{1}{n'+n}\times\sin(n't+nt+\epsilon'+\epsilon-\Pi)\right\},$$
$$+\frac{m'}{4}a^2a'n\Sigma B_{(i-1)}g\left\{\begin{array}{l}\frac{1}{i(n'-n)}\sin(i(n't-nt+\epsilon'-\epsilon)-\Pi)\\ -\frac{1}{i(n'-n)+2n}\sin\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-\Pi\}\end{array}\right\}$$

[and] [3] [4]

$$d\,q = \frac{m'}{2}\cdot\frac{a^2 n}{a'^2} g\left\{\frac{1}{n'+n}\cos\left(n't+nt+\epsilon'+\epsilon-\Pi\right)+\frac{1}{n'-n}\times\cos\left(n't-nt+\epsilon'-\epsilon-\Pi\right)\right\},$$

$$-\frac{m'}{4}a^2 a' n\Sigma B_{(i-1)} g\left\{\begin{array}{l}\frac{1}{i\left(n'-n\right)}\cos\left(i\left(n't-nt+\epsilon'-\epsilon\right)+\Pi\right)\\ +\frac{1}{i\left(n'-n\right)+2n}\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon-\Pi\right\}\end{array}\right\}.$$

530. The equations which determine the variations in the greater axes and mean motion show that these two elements are subject to very considerable periodic variations, depending on the configurations of the bodies, when the divisor $i\left(n'-n\right)+n$ or $i'n'-in$ is very small.

There is no instance of the mean motions of any two of the celestial bodies being so exactly commensurable as to have $i'n'-in=0,$ therefore the greater axes and mean motions have no secular inequalities, but in several instances this divisor is a very small fraction, and as a quantity is increased in value when divided by a fraction, the divisor $i'n'-in,$ and still more its square, increases the values of these periodic variations very much. For this reason the periodic variation in the mean motion is much greater than that in the greater axis, evidently arising from the double integration in the former.

531. It is unnecessary to add constant quantities to the preceding integrals, for they may be included in the elements of elliptical motion, which then become

$$a+a_{/},\ e+e_{/},\ v+v_{/},\ \epsilon+\epsilon_{/},\ p+p_{/},\ q+q_{/};$$

and in the troubled orbit they are

$$a+a_{/}+d\,a,\ e+e_{/}+d\,e,\ v+v_{/}+d\,v,\ \epsilon+\epsilon_{/}+d\,\epsilon,\ p+p_{/}+d\,p,\ q+q_{/}+d\,q.$$

Since $a_{/},\ e_{/},$ &c., $d\,a,\ d\,e,$ &c., are very small quantities of the order $m',\ a+a_{/},\ e+e_{/},$ &c., may be substituted in the latter quantities instead of *a*, *e*, &c., they will then be functions of the time and of the six constant quantities $a+a_{/},\ e+e_{/},$ &c.: so that the formulae of troubled motion in reality contain but six arbitrary constant quantities, as they ought to do. In order to determine $a_{/},\ e_{/},$ &c., suppose the perturbations of the planet m were required during a given interval of time. The quantities *a*, *e*, &c., are given by observation at the epoch when $t=0$ in the elliptical orbit, that is, assuming the disturbing force to be zero; but as $a_{/}+d\,a,\ e_{/}+d\,e,$ &c., arise entirely from the disturbing force, they must also be zero at the epoch; therefore, values of the arbitrary constant quantities $a_{/},\ e_{/},$ &c., are obtained from the equations[5]

$$a_{/}+d\bar{a}=0,\ e_{/}+d\bar{e}=0,\ v+d\bar{v}=0,\ \&c.,$$

$d\,\bar{a},\ d\,\bar{e},$ &c. being the values of $d\,a,\ d\,e,$ &c., at the epoch.

The effect of the disturbing forces upon each of the elliptical elements will be completely expressed by $a_{/}+d\bar{a}$, $e_{/}+d\bar{e}$, &c. during the time under consideration. Thus both the periodic and secular variations of the elements of the orbits are determined.

Notes

[1] A misplaced extra parenthesis in the fifth term *reads* $N_4 \cos\{i(n't-nt)+\in'-\in)+v-v'\}$ in the 1st edition.

[2] The first line factor $\sin(n't+nt+\in'+\in-\Pi)$ *reads* $\sin(n't+n't+\in'+\in-\Pi)$ in the 1st edition.

[3] The factor $\frac{1}{n'-n}\times\cos(n't-nt+\in'-\in-\Pi)$ in the 1st line *reads* $\frac{1}{n'-n}\times\cos(n't-nt+\in'-\in)-\Pi)$ in 1st ed.

[4] The factor $\frac{1}{i(n'-n)}\cos(i(n't-nt+\in'-\in)-\Pi)$ in the second line contains two errors in the 1st edition. It *reads* $\frac{1}{i(n'-n}\cos(i(n't-nt+\in'-\in)+\Pi)$ (published erratum in sign before Π).

[5] The 1st equation *reads* $a_{/}+da=0$ in the 1st edition.