BOOK II

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CHAPTER VI

SECULAR INEQUALITIES IN THE ELEMENTS OF THE ORBITS

Stability of the Solar System, with regard to the Mean Motions of The Planets and the greater axes of their Orbits

462. WHEN the squares of the disturbing masses are omitted, however far the approximation may be carried with regard to the eccentricities and inclinations, the general form of the series represented by R, in article 449, is

$$m'k \cdot \cos\{i'n't - int + c\} = R,$$

k and c are quantities consisting entirely of the elements of the orbits, k being a function of the mean distances, eccentricities, and inclinations, and c a function of the longitudes of the epochs of the perihelia and nodes. The differential of this expression, with regard to nt the mean motion of m, is

$$\mathrm{d}R = m'kindt \sin\{i'n't - int + c\}.$$

The expression dR always relates to the mean motion of m alone; when substituted in

$$da = 2a^2\mathrm{d}R,$$

it gives

$$da = 2a^2 m'ik \cdot ndt \cdot \sin\{i'n't - int + c\},$$

the integral of which is

$$da = \frac{2a^2 im'nk}{i'n' - in} \cdot \cos\{i'n't - int + c\}.$$

It is evident that if the greater axes of the orbits of the planets be subject to secular inequalities, this value of da must contain terms independent of the sines and cosines of the angular distances of the bodies from each other. But a must be periodic unless $i'n' - in = 0$; that is, unless the mean motions of the bodies m and m' be commensurable. Now the mean motions of no two bodies in the solar system are exactly commensurable, therefore $i'n' - in$ is in no case exactly zero; consequently the greater axes of the celestial bodies are not subject to secular inequalities; and on account of the equation $n = a^{-\frac{3}{2}}$, their mean motions are uniform.

Thus, when the squares and products of the masses m, m' are omitted, the differential $\mathrm{d}R$ does not contain any term proportional to the element of the time, however far the approximation may be carried with regard to the eccentricities and inclinations of the orbits, or which is the same thing, $\frac{\mathrm{d}R}{ndt}$ does not contain a constant term; for if it contained a term of the form $m'k$, then would $a = 2\int a^2 \,.\, \mathrm{d}R = 2a^2m'knt$, and $z = -3\iint andt \,.\, \mathrm{d}R$ would become

$$z = -3\iint an^2m'kdt^2 = -3an^2m'kt^2 ,$$

so that the greater axes would increase with the time, and the mean motion would increase with the square of the time, which would ultimately change the form of the orbits of the planets, and the periods of their revolutions. The stability of the system is so important, that it is necessary to inquire whether the greater axes and mean motions be subject to secular inequalities, when the approximation is carried to the squares and products of the masses.

463. The terms depending on the squares and products of the masses are introduced into the series R by the variation of the elements of the orbits, both of the disturbed and disturbing bodies. Hence, if δa, δe, &c. be the integrals of the differential equations of the elements in article 439, the variable elements will be $a+\delta a$, $e+\delta e$, &c. for m, and $a'+\delta a'$, $e'+\delta e'$, &c. for m'; and when these are substituted for a, e, a', e', &c. in the series R, it takes the form

$$R_/ = R+\delta R+\delta' R ;$$

and from what has been said, the greater axis and mean motion of m will not be affected by secular inequalities, unless the differential

$$\mathrm{d}R_/ = \mathrm{d}R + \mathrm{d}\,.\,\delta R + \mathrm{d}\,.\,\delta' R$$

contains a term that is not periodic.

[The term] dR is of the first order relatively to the masses, and has been proved in the preceding article not to contain a term that is not periodic. [The terms] $\mathrm{d}\,.\,\delta R$ and $\mathrm{d}\,.\,\delta' R$ include the squares and products of the masses; the first is the differential of δR with regard to the elements of the troubled planet m, and $\mathrm{d}\,.\,\delta' R$ is a similar function with regard to the disturbing body m'. It is proposed to examine whether either of these contains[1] a term that is not periodic, beginning with $\mathrm{d}\,.\,\delta R$.

464. The variation δR regards the elements of m alone, and is[2]

$$\delta R = \frac{dR}{da}\delta a + \frac{dR}{d\epsilon}\delta\epsilon + \frac{dR}{de}\delta e + \frac{dR}{d\varpi}\delta\varpi + \frac{dR}{dp}\delta p + \frac{dR}{dq}\delta q .$$

If the values in article 439, be put for δa, δe, &c. this expression becomes

$$\delta R = 2a^2\left\{\frac{dR}{da}\int\frac{dR}{d\epsilon}\cdot ndt - \frac{dR}{d\epsilon}\int\frac{dR}{da}\cdot ndt\right\}$$

$$+\frac{a\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left\{\frac{dR}{d\epsilon}\int\frac{dR}{de}\cdot ndt-\frac{dR}{de}\int\frac{dR}{d\epsilon}\cdot ndt\right\}$$
$$+\frac{a\sqrt{1-e^2}}{e}\left\{\frac{dR}{d\nu}\int\frac{dR}{de}\cdot ndt-\frac{dR}{de}\int\frac{dR}{d\nu}\cdot ndt\right\}$$
$$+\frac{a}{\sqrt{1-e^2}}\left\{\frac{dR}{dp}\int\frac{dR}{dq}\cdot ndt-\frac{dR}{dq}\int\frac{dR}{dp}\cdot ndt\right\}.$$

And its differential, according to the elements of the orbit of *m* alone, is obtained by suppressing the signs $\int$ introduced by the integration of the differential equations of the elements in article 439, which reduces this expression to zero; therefore to obtain $\mathrm{d}.dR$, it is sufficient to take the differential according to *nt* of those terms in dR that are independent of the sign $\int$.

When the series in article 449 is substituted for *R*, dR will take the form

$$P.f.Qdt-Q.f.Pdt.$$

Where *P* and *Q* represent a series of terms of the form[3]

$$k.\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(i'n't-int+c),$$

i' and *i* being any whole numbers positive or negative. Let[4]

$$k\cos(i'n't-int+c)$$

belong to *P*, and let $k'\cos(i'n't-int+c')$ be the corresponding term of *Q*, k, k', c, c', being constant quantities.

A term that is not periodic could only arise in

$$\mathrm{d}dR=\mathrm{d}\left\{P\int Qdt-Q\int Pdt\right\},$$

if it contained such an expression as

$$kk'\cos\{i'n't-int+c\}\cos\{i'n't-int+c'\}=\tfrac{1}{2}kk'\cos(c-c')+\tfrac{1}{2}kk'\cos\{2i'n't-2int+c+c'\};$$

or a similar product of the sines of the same angles. But when $k\cos(i'n't-int+c)$ is put for *P*, and $k'\cos(i'n't-int+c')$ for *Q*, $\mathrm{d}.dR$ becomes

$$\mathrm{d}.dR=kindt.\sin(i'n't-int+c).\int k'dt.\cos(i'n't-int+c')$$
$$-k'indt.\sin(i'n't-int+c').\int kdt.\cos(i'n't-int+c'),$$

which is equal to zero when the integrations are accomplished. Whence it way be concluded that $\mathrm{d}.dR$ is altogether periodic.

465. It now remains to determine whether the variation of the elements of the orbit of m' produces terms that are not periodic in $\mathrm{d}.d'R$. This cannot be demonstrated by the same process, because the function R, not being symmetrical relatively to the co-ordinates of m and m', changes its value in considering the disturbance of m' by m. Let R' be what R becomes with regard to the planet m' troubled by m; then

$$R' = m\left\{\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} - \frac{xx'+yy'+zz'}{r^3}\right\}$$

hence

$$R = \frac{m'}{m}R' + m'(xx'+yy'+zz')\left(\frac{1}{r^3}-\frac{1}{r'^3}\right);$$

and[5]

$$d'R = \frac{m'}{m}d'R' + m'd'\left\{(xx'+yy'+zz')\left(\frac{1}{r^3}-\frac{1}{r'^3}\right)\right\}.$$

If the differential of this equation according to d be periodic, so will $\mathrm{d}.d'R$. Now in consequence of the variations of the elements of the orbit of m,[6]

$$d'R' = \frac{dR'}{da'}da' + \frac{dR'}{de'}de' + \frac{dR'}{d\epsilon'}d\epsilon' + \frac{dR'}{dv'}dv' + \frac{dR'}{dp'}dp' + \frac{dR'}{dq'}dq'.$$

And as this expression with regard to the planet m' is in all respects similar to that of dR in the preceding article with regard to m, by the same analysis it may be proved that $\mathrm{d}.dR'$ is altogether periodic. Thus the only terms that are not periodic, must arise from the differential[7][8] of,

$$m'd'\{xx'+yy'+zz'\}\left(\frac{1}{r^3}-\frac{1}{r'^3}\right).$$

Let,[9]

$$m'\{xx'+yy'+zz'\}\left(\frac{1}{r^3}-\frac{1}{r'^3}\right) = L.$$

Then by article 346,

$$\frac{m'x}{r^3} = -\frac{m'}{S}\cdot\frac{d^2x}{dt^2} - \frac{mm'}{S}\cdot\frac{x}{r^3} + \frac{m'}{S}\left(\frac{dR}{dx}\right);$$

likewise

$$\frac{m'x'}{r'^3} = -\frac{m'}{S}\cdot\frac{d^2x'}{dt^2} - \frac{m'^2}{S}\cdot\frac{x'}{r'^3} + \frac{m'}{S}\left(\frac{dR'}{dx'}\right).$$

The co-ordinates $y,\ z,\ y',\ z',$ furnish similar equations. Thus,

$$L = \frac{m'}{S}\left\{\frac{d\left(xdx' - x'dx + ydy' - y'dy + zdz' - z'dz\right)}{dt^2}\right\} + N\,,$$

where

$$N = \frac{m'^2}{S}\left(\frac{xx' + yy' + zz'}{r'^3}\right) - \frac{mm'}{S}\left(\frac{xx' + yy' + zz'}{r^3}\right)$$
$$+\frac{m'}{S}\left\{x'\left(\frac{dR}{dx}\right) - x\left(\frac{dR'}{dx'}\right) + y'\left(\frac{dR}{dy}\right) - y\left(\frac{dR'}{dy'}\right) + z'\left(\frac{dR}{dz}\right) - z\left(\frac{dR'}{dz'}\right)\right\}.$$

If N be omitted at first,

$$\mathrm{d}\cdot L = \frac{m'}{S}\cdot d\left\{\frac{d\left(x'dx - xdx' + y'dy - ydy' + z'dz - zdz'\right)}{dt^2}\right\}.$$

466. The elliptical values of the co-ordinates being substituted, every term must be periodic. For example, if

$$x = a\cdot\cos\left(nt + \epsilon - v\right) \qquad x' = a'\cdot\cos\left(n't + \epsilon' - v'\right)$$

[then]

$$\frac{x'dx - xdx'}{dt} = \tfrac{1}{2}aa'\left(n - n'\right).\sin\left\{n't - nt + \epsilon' - \epsilon - v' + v\right\};$$

a quantity that must be periodic unless $n't - nt = 0,$ which never can happen, because the mean motions of no two bodies in the solar system are exactly commensurable; but even if a term that is not periodic were to occur, it would vanish in taking the second differential; and as the same thing may be shown with regard to the other products

$$y'dy - ydy' \text{ [and] } z'dz - zdz',$$

$\mathrm{d}L$ is a periodic function. With regard to the term $\mathrm{d}L = \mathrm{d}N,$ if the elliptical values of the co-ordinates of m and m' be substituted, it will readily appear that this expression is periodic, for the equations of the elliptical motion of m and $m',$ in article 365, give

$$\frac{xx' + yy' + zz'}{r^3} = -\frac{x'd^2x + y'd^2y + z'd^2z}{\left(S + m\right)dt^2}\,,$$
$$\frac{xx' + yy' + zz'}{r'^3} = -\frac{x'd^2x + y'd^2y + z'd^2z}{\left(S + m'\right)dt^2}\,;$$

so that the function *N* becomes

$$N = -\frac{m'^2}{S(S+m)}\left\{\frac{xd^2x' + yd^2y' + z'd^2z}{dt^2}\right\} + \frac{mm'}{S(S+m)}\left(\frac{x'd^2x + y'd^2y + z'd^2z}{dt^2}\right)$$
$$+\frac{m'}{S}\left\{x'\left(\frac{dR}{dx}\right) - z\left(\frac{dR'}{dx'}\right) + y'\left(\frac{dR}{dy}\right) - y\left(\frac{dR'}{dy'}\right) + z'\left(\frac{dR}{dz}\right) - z\left(\frac{dR'}{dz'}\right)\right\}.$$

467. From what has been said, it will readily appear that the terms of this expression, consisting of the products $x'd^2x$, xd^2x', &c. &c., are periodic when the elliptical values are substituted for the co-ordinates, and their differentials.

468. The last term of the value of *N* is also periodic; for, if the elliptical values of the co-ordinates of *m* and m', be put in *R*, it may be developed into a series of cosines of the multiples of the arcs *nt* and $n't$, and the differential may be found by making *R* vary with regard to the quantities belonging to *m* alone; hence this differential may contain the sines and cosines of the multiples of *nt*, but no sine or cosine of $n't$ alone; and as[10]

$$x' = a'\cos(nt + \in' - v'),$$

the mean motions nt, $n't$, never vanish from $x'\left(\frac{dR}{dx}\right)$, which is consequently periodic; and as the same may be demonstrated for each of the products

$$x\left(\frac{dR'}{dx'}\right),\ y'\left(\frac{dR}{dy}\right), \text{ \&c. \&c.,}$$

not only *N* but its differential are periodic, and consequently $\mathrm{d}.d'R$.

Thus it has been proved that when the approximation is carried to the squares and products of the masses, the expression

$$\mathrm{d}R_{/} = \mathrm{d}R + \mathrm{d}.d\,R + \mathrm{d}.d'R$$

relatively to the variations of the mean motions of the two planets *m* and m' is periodic.

469. These results would be the same whatever might be the number of disturbing bodies; for m'' being a second planet disturbing the motion of *m*, it would add to *R* the term

$$\frac{m''}{\sqrt{(x''-x)^2 + (y''-y)^2 + (z''-z)^2}} - \frac{m''(xx'' + yy'' + zz'')}{r''^3}.$$

The variation of the co-ordinates of [11] m and m'' resulting from the reciprocal action of these two planets, would produce terms multiplied by mm'' and m''^2 in the variation of R; and by the preceding analysis it follows that all the terms in $\mathrm{d}\,.d''R$ are periodic. $d''R$ relates to the variation of the elements of the orbit of m''.

The variations of the co-ordinates of m' arising from the action of m'' on m', will cause a variation in the part of R depending on the action of m' on m represented by

$$\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}-\frac{m'(xx'+yy'+zz')}{r'^3}$$

There will arise terms in R, multiplied by $m'm''$, which will be functions of nt, $n't$, $n''t$, when substitution is made of the elliptical values of the co-ordinates; and as the mean motions cannot destroy each other, these terms will only produce periodic terms in dR. Should there be any terms independent of the mean motion nt in the development of R, they will vanish by taking the differential dR. And as terms depending on nt alone will have the form $m'm''.\mathrm{d}P$, P being a function of the elliptical co-ordinates of m,[12] there will arise in $\int \mathrm{d}\,.R$ terms of the form $m'm''\int \mathrm{d}P = m'm''.P$, since d$P$ is an exact differential. These terms will then be of the second order after integration, and such terms are omitted in the value of this function.

The variation of the co-ordinates x, y, z, produced by the action of m'' on m only introduce into the preceding part of R terms multiplied by $m'm''$ and functions of the three angles nt, $n't$, $n''t$; and as these three mean motions cannot destroy each other, there can only be periodic terms in dR. The terms depending on nt alone, only produce periodic terms of the order $m'm''$ in dR.

The same may be proved with regard to the part of R depending on the action of m'' on m.

470. Hence whatever may be the number of disturbing bodies, when the approximation includes the squares and products of the masses, the variation of the elliptical elements of the disturbed and disturbing planets only produce periodic terms in dR.

471. Now the variation of

$$z = -3\iint an\,dt\,.\,\mathrm{d}R$$

is

$$dz = -3an\iint dt\,.\,\mathrm{d}\,.\,dR + 3a^2\iint\left(ndt\,.\,\mathrm{d}R\,.\int \mathrm{d}R\right).$$

It was proved in article 464 that $\mathrm{d}dR = 0$ in considering only secular quantities of the order of the squares of the masses. It is easy to see from the form of the series R that $\mathrm{d}R\int \mathrm{d}R = 0$ with regard to these quantities, consequently the variation of the mean motion of a planet cannot contain any secular inequality of the first or second order with regard to the disturbing forces that can become sensible in the course of ages, whatever the number of planets may be that trouble its motion. And as $da = 2a^2 \mathrm{d}R$ becomes

$$d\,a = \left\{2a^2 \int \mathrm{d}R + 8a^3 \int \left(\mathrm{d}R \int \mathrm{d}R\right)\right\},$$

by the substitution of $\left(a + d\,a^2\right)$ for a^2, $d\,a$ cannot contain a secular inequality if dz does not contain one.

472. It therefore follows, that when periodic inequalities are omitted as well as the quantities of the third order with regard to the disturbing forces, the *mean motions* of the planets, and the *greater axes* of their orbits, are *invariable*.

The whole of this analysis is given in the Supplement to the third volume of the *Mécanique Céleste*; but that part relating to the second powers of the disturbing forces is due to M. Poisson.[13]

Differential Equations of the Secular Inequalities in the Eccentricities, Inclinations, Longitudes of the Perihelia and Nodes, which are the annual and sidereal variations of these four elements

473. That part of the series *R*, in article 449, which is independent of periodic inequalities, is found by making $i = 0$, for then[14]

$$\sin i\left(n't - nt + \in' - \in\right) = 0,$$
$$\cos i\left(n't - nt + \in' - \in\right) = 1;$$

and if the differences of $A_0\, A_1$ with regard to a' be eliminated by their values in article 458, the series *R* will be reduced to

$$F = \frac{m'}{2}A_0 + \frac{m'}{4}\left\{a\left(\frac{dA_0}{da}\right) + \tfrac{1}{2}a^2\left(\frac{d^2A_0}{da^2}\right)\right\}\left(e^2 + e'^2\right)$$
$$+\frac{m'}{2}\left\{A_1 - a\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right)\right\}ee'\cos\left(v' - v\right)$$
$$-\frac{m'}{8}a\,a'B_1 g^2$$

But the formulae in articles 456 and 457 give

$$a\left(\frac{dA_0}{da}\right) + \tfrac{1}{2}a^2\left(\frac{d^2A_0}{da^2}\right) = -\frac{3aa'.S'}{2\left(a'^2 - a^2\right)^2}$$

$$A_1 - a\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right) = \frac{3\left(aa'S + \left(a^2 + a'^2\right)S'\right)}{\left(a'^2 - a^2\right)}$$

$$aa'B_1 = -\frac{3aa'.S'}{\left(a'^2 - a^2\right)^2};$$

consequently

$$F = \frac{m'}{2}A_0 - \frac{3m'.aa'.S'}{2.4.\left(a'^2 - a^2\right)^2}\cdot\left\{e^2 + e'^2 - (p' - p)^2 - (q' - q)^2\right\}$$

$$+\frac{3m'\left(a'a.S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot ee'.\cos(v' - v);$$

for by article 444

$$g^2 = (p' - p)^2 + (q' - q)^2,$$

whence

$$\frac{dF}{dv} = \frac{3m'\left(a'a.S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot ee'.\sin(v' - v)$$

$$\frac{dF}{de} = -\frac{3m'aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot e + \frac{3.m'\left(aa'S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot e'.\cos(v' - v)$$

$$\frac{dF}{dp} = -\frac{3m'.aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot(p' - p)$$

$$\frac{dF}{dq} = -\frac{3m'.aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot(q' - q).$$

474. When the squares of the eccentricities are omitted, the differential equations in article 441 become[15]

$$\frac{de}{dt} = -\frac{an}{e}\frac{dF}{dv}; \quad \frac{dv}{dt} = \frac{an}{e}\frac{dF}{de};$$

$$\frac{dp}{dt} = an\cdot\frac{dF}{dq}; \quad \frac{dq}{dt} = -an\cdot\frac{dF}{dp}.$$

If the differentials of F, according to the elements, be substituted in these, and if to abridge[16]

$$-\frac{3m'.na^2a'S'}{4\left(a'^2 - a^2\right)^2} = (0.1);$$

$$-\frac{3m'\cdot an\cdot\left(aa'S+\left(a^2+a'^2\right)S'\right)}{2\left(a'^2-a^2\right)}=\boxed{0.1};$$

they become

$$\begin{aligned}
\frac{de}{dt}&=\boxed{0.1}\,e'\sin\left(v'-v\right)\\
\frac{dv}{dt}&=(0.1)-\boxed{0.1}\,\frac{e'}{e}\cos\left(v'-v\right)\\
\frac{dp}{dt}&=-(0.1)\left(q-q'\right)\\
\frac{dq}{dt}&=+(0.1)\left(p-p'\right).
\end{aligned}\tag{127}$$

475. But $\tan f=\sqrt{p^2+q^2}$ and $\tan q=\frac{p}{q}$, and when the squares of the inclinations are omitted $\cos q=1$, hence

$$df=dp\sin q+dq\cos q;\quad dq=\frac{dp\cos q-dq\sin q}{\tan f};$$

and substituting the preceding values of *dp*, *dq*, the variations in the inclinations and longitude of the node are,[17]

$$\begin{aligned}
\frac{df}{dt}&=(0.1)\,.\tan f\,.\sin\left(q-q'\right)\\
\frac{dq}{dt}&=-(0.1)+(0.1)\cdot\frac{\tan f'}{\tan f}\cdot\cos\left(q-q'\right).
\end{aligned}$$

476. The preceding quantities are the secular variations in the orbit of *m* when troubled by m' alone, but all the bodies in the system act simultaneously on the planet *m*, and whatever effect is produced in the elements of the orbit of *m* by the disturbing planet m', similar effects will be occasioned by the disturbing bodies[18] m'', m''', &c. Hence, as the change produced by m' in the elements of the orbit of *m* are expressed by the second terms of the preceding equations, it is only necessary to add to them a similar quantity for each disturbing body, in order to have the whole action of the system on *m*.

The expressions (0.1), $\boxed{0.1}$ have been employed to represent the coefficients relative to the action of m' on *m*; for quantities relative to *m* which has no accent, are represented by 0; and those relating to m' which has one accent, by 1; following the same notation, the coefficients relative to the action of m'' on *m* will be (0.2), $\boxed{0.2}$; those relating to m''' on *m* by (0.3), $\boxed{0.3}$; and so on. Therefore the secular action of m'' in disturbing the elements of the orbit of *m* will be

$$\boxed{0.2}\, e''\sin(v''-v); \quad (0.2)-\boxed{0.2}\frac{e''}{e}\cos(v''-v)$$
$$(0.2)\tan f\sin(q-q''); \quad -(0.2)+(0.2)\frac{\tan f''}{\tan f}\cos(q-q'').$$

477. Therefore the differential equations of the secular inequalities of the elements of the orbit of m, when troubled by the simultaneous action of all the bodies in the system, are

$$\begin{aligned}
\frac{de}{dt} &= \boxed{0.1}\, e'\sin(v'-v)+\boxed{0.2}\, e''\sin(v''-v)+\boxed{0.3}\, e'''\sin(v'''-v)+\&\text{c.}\\
\frac{dv}{dt} &= (0.1)+(0.2)+\&\text{c.}-\boxed{0.1}\,\frac{e'}{e}\cos(v'-v)-\boxed{0.2}\,\frac{e''}{e}\cos(v''-v)-\&\text{c.}\\
\frac{df}{dt} &= (0.1)\tan f'\sin(q-q')+(0.2)\tan f''\sin(q-q'')+\&\text{c.}\\
\frac{dq}{dt} &= -\{(0.1)+(0.2)+\&\text{c.}\}+(0.1)\frac{\tan f'}{\tan f}\cos(q-q')+(0.2)\frac{\tan f''}{\tan f}\cos(q-q'')+\&\text{c.}
\end{aligned} \qquad \textbf{(128)}$$

478. All the quantities in these equations are determined by observation for a given epoch assumed as the origin of the time, and when integrated, or (which is the same thing) multiplied by t, they give the annual variation in the elements of the orbit of a planet, on account of the immense periods of the secular inequalities, which admit of one year being regarded as an infinitely short time in which the elements e, v, &c., may be supposed to be constant.

479. It is evident that the secular variations in the elements of the orbits of m', m'', m''', &c., will be obtained from the preceding equations, if every thing relating to m be changed into the corresponding quantities relative to m', and the contrary, and so for the other bodies. Thus the variation in the elements of m', m'', &c., from the action of all the bodies in the system, will be

$$\begin{aligned}
\frac{de'}{dt} &= \boxed{1.0}\,.e.\sin(v-v')+\boxed{1.2}.\, e''.\sin(v''-v')+\&\text{c.}\\
\frac{de''}{dt} &= \boxed{2.0}\,.e.\sin(v-v'')+\boxed{2.1}.\, e'.\sin(v'-v'')+\&\text{c.}\\
&\qquad\qquad\qquad \&\text{c.} \qquad \&\text{c.}\\
\frac{dv'}{dt} &= (1.0)+(1.2)+\&\text{c.}-\boxed{1.0}\,.\frac{e}{e'}.\cos(v-v')-\boxed{1.2}\,.\frac{e''}{e'}\cos(v''-v').-\&\text{c.}\\
\frac{dv''}{dt} &= (2.0)+(2.1)+\&\text{c.}-\boxed{2.0}\,.\frac{e}{e''}\cos(v-v'')-\boxed{2.1}\,.\frac{e'}{e''}\cos(v'-v'').-\&\text{c.}\\
&\qquad\qquad\qquad \&\text{c.} \qquad \&\text{c.}\\
\frac{df'}{dt} &= (1.0).\tan f.\sin(q'-q)+(1.2).\tan f''.\sin(q'-q'')+\&\text{c.}\\
\frac{df''}{dt} &= (2.0).\tan f.\sin(q''-q)+(2.1).\tan f'.\sin(q''-q')+\&\text{c.}
\end{aligned} \qquad \textbf{(129)}$$

$$\frac{dq'}{dt} = -\{(1.0)+(1.2)+\&c.\} + (1.0)\cdot\frac{\tan f}{\tan f'}\cdot\cos(q'-q) + (1.2)\frac{\tan f''}{\tan f'}\cdot\cos(q'-q'') + \&c.$$
$$\frac{dq''}{dt} = -\{(2.0)+(2.1)+\&c.\} + (2.0)\cdot\frac{\tan f}{\tan f''}\cos(q''-q) + (2.1)\cdot\frac{\tan f}{\tan f''}\cos(q''-q') + \&c.$$

&c. &c.

As these quantities do not contain the mean longitude, nor its sines or cosines, they depend on the configuration of the orbits only.

Approximate Values of the Secular Variations in these four Elements in Series, ascending according to the powers of the Time

480. The annual variations in the elements are readily obtained from these formulae; but as the secular inequalities vary so slowly that they may be assumed to vary as the time for a great many centuries without sensible error, series may be formed, whence very accurate values of the elements may be computed for at least a thousand years before and after the epoch. Let the eccentricity be taken as an example. With the given values of the masses and mean longitudes of the perihelia determined by observation, let a value of $\frac{de}{dt}$, the variation in the eccentricity, be computed from the preceding equation for the epoch, say 1750, and another for 1950. If the latter be represented by $\left(\frac{de}{dt}\right)$, and the former by $\left(\frac{d\bar{e}}{dt}\right)$, then

$$\left(\frac{de}{dt}\right) - \frac{d\bar{e}}{dt} = 200\cdot\frac{d^2\bar{e}}{dt^2}; \text{ or, } \left(\frac{de}{dt}\right) = \left(\frac{d\bar{e}}{dt}\right) + 200\cdot\frac{d^2\bar{e}}{dt^2}$$

the quantities $\frac{d\bar{e}}{dt}$, $\frac{d^2\bar{e}}{dt^2}$, being relative to the year 1750. Hence, $\bar{e}$ being the eccentricity of any orbit at that epoch, the eccentricity e at any other assumed time t, may be found from[19]

$$e = \bar{e} + \frac{d\bar{e}}{dt}\cdot t + \tfrac{1}{2}\cdot\frac{d^2\bar{e}}{dt^2}t^2 + \&c.$$

with sufficient accuracy for 1,000 or 1,200 years before and after 1750.

In the same manner all the other elements may be computed from

$$
\begin{aligned}
v &= \bar{v} + \frac{d\bar{v}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{v}}{dt^2}\cdot t^2 + \&\text{c.} \\
f &= \bar{f} + \frac{d\bar{f}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{f}}{dt^2}\cdot t^2 + \&\text{c.} \\
q &= \bar{q} + \frac{d\bar{q}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{q}}{dt^2}\cdot t^2 + \&\text{c.} \\
g &= \bar{g} + \frac{d\bar{g}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{g}}{dt^2}\cdot t^2 + \&\text{c.} \\
\Pi &= \bar{\Pi} + \frac{d\bar{\Pi}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{\Pi}}{dt^2}\cdot t^2 + \&\text{c.}
\end{aligned}
\qquad \textbf{(130)}
$$

For as and $\bar{f}$ and $\bar{q}$ are given by observation, $\bar{g}$ and $\bar{\Pi}$, which are functions of them, may be found. All the quantities in these equations are relative to the epoch.

These expressions are sufficient for astronomical purposes; but as very important results may be deduced from the finite values of the secular variations, the integrals of the preceding differential equations must be determined for any given time.

Finite Values of the Differential Equations relative to the eccentricities and longitudes of the Perihelia

481. Direct integration is impossible in the present state of analysis, but the differential equations in question may be changed into linear equations capable of being integrated by the following method of Lagrange. Let

$$
\begin{aligned}
h &= e\sin v & \qquad l &= e\cos v \\
h' &= e'\sin v' & \qquad l' &= e'\cos v', \\
&\&\text{c.} & &\&\text{c.}
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{dh}{dt} &= \frac{de}{dt}\sin v + \frac{dv}{dt}\cdot e\cos v\,, \\
\frac{dl}{dt} &= \frac{de}{dt}\cos v - \frac{dv}{dt}\cdot e\sin v\,;
\end{aligned}
$$

and substituting the differentials in article 477, the result will be[20]

$$
\begin{aligned}
\frac{dh}{dt} &= \{(0.1)+(0.2)+\&\text{c.}\}\,l - \boxed{0.1}\,l' - \boxed{0.2}\,l'' - \boxed{0.3}\,l''' - \&\text{c.} \\
\frac{dl}{dt} &= -\{(0.1)+(0.2)+\&\text{c.}\}\,h + \boxed{0.1}\,h' + \boxed{0.2}\,h'' + \boxed{0.3}\,h''' + \&\text{c.}
\end{aligned}
\qquad \textbf{(131)}
$$

likewise
$$
\frac{dh'}{dt} = \{(0.1)+(0.2)+\&\text{c.}\}\,l' - \boxed{0.1}\,l - \boxed{1.2}\,l'' - \boxed{0.3}\,l''' - \&\text{c.}
$$

$$\frac{dl'}{dt} = -\{(0.1)+(1.2)+\&\text{c.}\}h' + \boxed{0.1}\, h + \boxed{1.2}\, h'' + \boxed{1.3}\, h''' + \&\text{c.}$$
$$\&\text{c.} \qquad \&\text{c.}$$

It is obvious that there must be twice as many such equations, and as many terms in each, as there are bodies in the system.

482. The integrals of these equations will be obtained by making

$$\begin{aligned} h &= N\sin(gt+x) \qquad & l &= N\cos(gt+x) \\ h' &= N'\sin(gt+x) \qquad & l' &= N'\cos(gt+x), \\ &\&\text{c.} & &\&\text{c.} \end{aligned}$$

It is easy to see why these quantities take this form, for if $h'=0,\ h''=0,$ &c., $l=0,$ $l'=0,$ &c., then

$$\frac{dh}{dt} = (0.1)\,l;\ \frac{dl}{dt} = -(0.1)\,h.$$

Let[21]

$$\frac{dh}{dt} = gl;\ \frac{dl}{dt} = -gh,$$

but

$$\frac{d^2h}{dt^2} = g\,\frac{dl}{dt},$$

therefore

$$\frac{d^2h}{dt^2} + g^2h = 0.$$

And by article 214 $h = N\sin(gt+x)$, N and x being arbitrary constant quantities. In the same manner $l = N\cos(gt+x)$.

483. If the preceding values of $h,\ h',\ h''$, &c., $l,\ l',\ l''$, &c., and their differentials be substituted in equations (131), the sines and cosines vanish, and there will result a number of equations,

$$\begin{aligned} Ng &= \{(0.1)+(0.2)+(0.3)+\&\text{c.}\}N - \boxed{0.1}\, N' - \boxed{0.2}\, N'' - \&\text{c.} \\ N'g &= \{(1.0)+(1.2)+(1.3)+\&\text{c.}\}N' - \boxed{1.0}\, N - \boxed{1.2}\, N'' - \&\text{c.} \\ &\&\text{c.} \qquad \&\text{c.} \end{aligned} \qquad \textbf{(132)}$$

equal to the number of quantities $N,\ N',\ N''$, &c., consequently equal to the number of bodies in the system; hence, if $N',\ N'',\ N'''$, &c., be eliminated, N will vanish, and will therefore remain indeterminate, and there will result an equation in g only, the degree of which will be equal to the number of bodies $m,\ m',\ m''$, &c. The roots of this equation way be represented by g, g_1, g_2, &c., which are the mean secular motions of the perihelia of the orbits of $m,\ m',\ m''$, &c., and are functions of the known quantities (0.1), $\boxed{0.1}$, (1.0), $\boxed{1.0}$, &c., only. When successively substituted in equations (132), these equations will only contain the indeterminate quantities N, $N',\ N''$, &c.; but it is clear, that for each root of g, quantities $N,\ N',\ N''$, &c., will have different values. Therefore let $N,\ N',\ N''$, &c., be their values corresponding to the root g; N_1, $N_1',\ N_1''$, &c., those corresponding to the root g_1;[22] $N_2,\ N_2',\ N_2''$, &c., those arising from the substitution of g_2, &c. &c.; and as the complete integral of a differential linear equation is the sum of the particular equations, the integrals of (131) are

$$\begin{aligned} h &= N\sin(gt+x) + N_1\sin(g_1t+x_1) + N_2\sin(g_2t+x_2) + \&c. \\ h' &= N'\sin(gt+x) + N_1'\sin(g_1t+x_1) + N_2'\sin(g_2t+x_2) + \&c. \\ &\qquad \&c. \qquad \&c. \\ l &= N\cos(gt+x) + N_1\cos(g_1t+x_1) + N_2\cos(g_2t+x_2) + \&c. \\ l' &= N'\cos(gt+x) + N_1'\cos(g_1t+x_1) + N_2'\cos(g_2t+x_2) + \&c. \\ &\qquad \&c. \qquad \&c. \end{aligned} \tag{133}$$

for each term contains two arbitrary quantities $N,\ x;\ N_1,\ x_1$, &c.

484. Since each term of the equations (132) has one of the quantities $N,\ N'$, &c., for coefficient, these equations will only give values of the ratios

$$\frac{N'}{N};\ \frac{N''}{N'};\ \&c.,$$

so that for each of the roots by g, g_1, g_2, &c., one of the quantities N, N_1, N_2, &c., will remain indeterminate.

To show how these are determined, it must be observed that in the expression

$$\boxed{0.1} = -\frac{3m'.an\left(aa'S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}$$

of article 474, S and S' are the coefficients of the first and second terms of the development of[23]

$$\left(a^2 - 2aa'\cos b + a'^2\right)^{\frac{1}{2}},$$

which remain the same when a' is put for a; and the contrary, that is to say, whether the action of m' on m be considered, or that of m on m'. Hence if $m,\ n'$, and a', be put for $m',\ n$, and a,

$$\boxed{0.1} = -\frac{3m\,.a'n'\left(a a' S + \left(a'^2 + a^2\right) S'\right)}{2\left(a'^2 - a^2\right)^2}$$

consequently[24]

$$\boxed{0.1}\ .m.n'a' = \boxed{1.0}\ .m'.na\,.$$

It is also evident that

$$(0.1)\,m\,.\,n'a' = (1.0)\,m'.\,na\,.$$

But if the mass of the planet be omitted in comparison of that of the sun considered as the unit,

$$n^2 = \frac{1}{a^3};\ n'^2 = \frac{1}{a'^3},\ \&c.;$$

therefore

$$\boxed{0.1}\,m\sqrt{a} - \boxed{1.0}\,m'\sqrt{a'} = 0\,,$$
$$\boxed{0.2}\,m\sqrt{a} - \boxed{2.0}\,m''\sqrt{a''} = 0\,,$$
$$\&c. \qquad \&c.$$
$$(0.1)\,m\sqrt{a} - (1.0)\,m'\sqrt{a'} = 0\,,$$
$$(0.2)\,m\sqrt{a} - (2.0)\,m''\sqrt{a''} = 0\,,$$
$$\&c. \qquad \&c.$$

485. Now let those of equations (131) that give

$$\frac{dh}{dt},\ \frac{dh'}{dt},\ \&c.,$$

be respectively multiplied by

$$Nm\sqrt{a},\ N'm'\sqrt{a'},\ N''m''\sqrt{a''},\ \&c.;$$

then, in consequence of equations (132), and the preceding relations, it will be found that

$$N\frac{dh}{dt}m\sqrt{a} + N'\frac{dh'}{dt}m'\sqrt{a'} + N''\frac{dh''}{dt}m''\sqrt{a''} + \&c.$$
$$= g\left\{Nlm\sqrt{a} + N'l'm'\sqrt{a'} + N''l''m''\sqrt{a''} + \&c.\right\};$$

if the preceding values of h, h', h'', &c., l, l', &c., be put in this, a comparison of the coefficients of like cosines gives[25]

$$0 = NN_1m\sqrt{a} + N'N_1'm'\sqrt{a'} + N''N_1''m''\sqrt{a''} + \&c.$$
$$0 = NN_2m\sqrt{a} + N'N_2'm'\sqrt{a'} + N''N_2''m''\sqrt{a''} + \&c.$$

Again, if the values of $h,\ h',\ h''$, &c., in equations (133) be respectively multiplied by

$$Nm\sqrt{a},\ N'm'\sqrt{a'},\ \text{\&c.}$$

They give

$$\begin{aligned}&Nmh\sqrt{a}+N'm'h'\sqrt{a'}+N''m''h''\sqrt{a''}+\text{\&c.}=\\&\left\{N^2m\sqrt{a}+N'^2m'\sqrt{a'}+N''^2m''\sqrt{a''}+\text{\&c.}\right\}\sin\left(gt+x\right),\end{aligned} \tag{134}$$

in consequence of the preceding relations.

By the same analysis the values of $l,\ l',\ l''$, &c., give

$$\begin{aligned}&Nml\sqrt{a}+N'm'l'\sqrt{a'}+N''m''l''\sqrt{a''}+\text{\&c.}=\\&\left\{N^2m\sqrt{a}+N'^2m'\sqrt{a'}+N''^2m''\sqrt{a''}+\text{\&c.}\right\}\cos\left(gt+x\right).\end{aligned}$$

The eccentricities of the orbits of the planets, and the longitudes of their perihelia, are known by observation at the epoch, and if these be represented by $\bar{e},\ \bar{e}'$, &c., $\bar{v},\ \bar{v}'$, &c. by article 481,

$$\begin{aligned}h&=\bar{e}\sin\bar{v},\ h'=\bar{e}'\sin\bar{v}',\ \&c.,\\ l&=\bar{e}\cos\bar{v},\ l'=\bar{e}'\cos\bar{v}',\ \&c.;\end{aligned}$$

therefore $h,\ h'$, &c., $l,\ l'$, &c., are given at that period. And if it be taken as the origin of the time $t=0$, and the preceding equations give[26]

$$\tan x=\frac{N.\bar{e}\sin\bar{v}\ .m\sqrt{a}+N'.\bar{e}'\sin\bar{v}'.m'\sqrt{a'}+\&c.}{N.\bar{e}\cos\bar{v}\ .m\sqrt{a}+N'.\bar{e}'\cos\bar{v}'.m'\sqrt{a'}+\&c.}.$$

But, for the root g, the equations (132) give

$$N'=CN,\ N''=C'N,\ N'''=C''N,\ \text{\&c.},$$

$C,\ C',\ C''$ being constant and given quantities; therefore

$$\tan x=\frac{\bar{e}\sin\bar{v}\ .m\sqrt{a}+C\ .\bar{e}'\sin\bar{v}'.m'\sqrt{a'}+\&c.}{\bar{e}\cos\bar{v}\ .m\sqrt{a}+C\ .\bar{e}'\cos\bar{v}'.m'\sqrt{a'}+\&c.}.$$

If these values of $N',\ N''$, &c., be eliminated from equation (134), it gives

$$N = \frac{\bar{e} \sin \bar{v}\, m\sqrt{a} + C\bar{e}' \sin \bar{v}' m' \sqrt{a'} + \&c.}{\left\{m\sqrt{a} + C^2 m' \sqrt{a'} + C'^2 m'' \sqrt{a''} + \&c.\right\} \sin x}.$$

Thus $\tan x$ and N are determined, all the remaining coefficients N', N'', &c., may be computed from equations (132), for the root g.

In this manner the indeterminate quantities belonging to the other roots g_1, g_2, &c., may be found. Thus the equations (133) are completely determined, whence the eccentricities of the orbits and the longitudes of their perihelia may be found for any instant $\pm t$, before or after the epoch.

486. The roots g, g_1, g_2, &c., express the mean secular motions of the perihelia, in the same manner that n represents the mean motion of a planet.

For example, the periodic time of the earth is about $365\frac{1}{4}$ days; hence $n = \frac{360^0}{365\frac{1}{4}}$, which is the mean motion of the earth for a day, and nt is its mean motion for any time t. The perihelion of the terrestrial orbit moves through 360^0 in 113,270 years nearly; hence, for the earth,[27]

$$g = \frac{360^0}{113{,}270} = 11''.44$$

in a century; and gt is the mean motion for any time t so that $nt + \in$ being the mean longitude of a planet, $gt + x$ is the mean longitude of its perihelion at any given time.

487. The equations (133), as well as observation, concur in proving that the perihelia have a motion in space, and that the eccentricities vary slowly. As, however, that variation might in process of time alter the nature of the orbits so much as to destroy the stability of the system, it is of the greatest importance to inquire whether these variations are unlimited, or if limited, what their extent is.

Stability of the Solar System with regard to the Form of the Orbits

488. Because

$$h = e \sin v\,,\quad l = e \cos v\,,\quad e^2 = h^2 + l^2;$$

and in consequence of the values of h and l in equations (133), the square of the eccentricity of the orbit of m becomes[28]

$$e^2 = N^2 + N_1^2 + N_2^2 + \&c. + 2NN_1 \cos\left\{(g_1 - g)t + x_1 - x\right\} + 2NN_2 \cos\left\{(g_2 - g)t + x_2 - x\right\} + \&c. \qquad \textbf{(135)}$$

When the roots $g,\ g_1$, &c., are all real and unequal, the cosines in this expression will oscillate between fixed limits, and e^2 will always be less than[29]

$$\left(N+N_1+N_2+\&\text{c.}\right)^2 = N^2+N_1^{\ 2}+\&\text{c.}+2NN_1+2NN_2+\&\text{c.}$$

taken with the same sign, for it could only obtain that maximum if

$$\left(g_1-g\right)t+x_1-x=0,\ \ \left(g_2-g\right)t+x_2-x=0,+\&\text{c.}$$

which could never happen unless the time were to vanish; that is, unless

$$g_1-g=0,\ \ g_2-g=0,\ \&\text{c.};$$

thus, if $g,\ g_1,\ g_2$, &c., be real and unequal, the value of e^2 will be limited.

489. If however any of these roots be imaginary or equal, they will introduce circular arcs or exponentials into the values of $h,\ h'$, &c., $l,\ l'$, &c.; and as these quantities would then increase indefinitely with the time, the eccentricities would no longer be confined to fixed limits, but would increase till the orbits of the planets, which are now nearly circular, become very eccentric.

The stability of the system therefore depends on the nature of the roots $g,\ g_1,\ g_2$, &c.: however it is easy to prove that they will all be real and unequal, if all the bodies $m,\ m',\ m''$, &c., in the system revolve in the same direction.

490. For that purpose let the equations

$$\frac{de}{dt}=\boxed{0.1}\,e'\sin\left(v'-v\right)+\boxed{0.2}\,e''\sin\left(v''-v'\right)+\&\text{c.}$$
$$\frac{de'}{dt}=\boxed{1.0}\,e\sin\left(v-v'\right)+\boxed{1.2}\,e''\sin\left(v'-v''\right)+\&\text{c.}$$
$$\&\text{c.}\qquad\&\text{c.}$$

be respectively multiplied by

$$me\sqrt{a},\ m'e'\sqrt{a'},\ m''e''\sqrt{a''},\ \&\text{c.},$$

and added; then in consequence of the relations in article 484, and because

$$\sin\left(v-v'\right)=-\sin\left(v'-v\right)$$
$$\sin\left(v-v''\right)=-\sin\left(v''-v\right),\ \ \&c.\ \&c.,$$

the sum will be

$$0=ede\,.\,m\sqrt{a}+e'de'.m'\sqrt{a'}+e''de''\,.\,m''\sqrt{a''}+\&\text{c.}$$

and as the greater axes of the orbits are constant, its integral is

$$e^2 m\sqrt{a} + e'^2 m'\sqrt{a'} + e''^2 m''\sqrt{a''} + \&\text{c.} = C\,. \qquad \textbf{(136)}$$

491. The radicals $\sqrt{a}$, $\sqrt{a'}$, &c., must all have the same sign if the planets revolve in the same direction; since by Kepler's law they depend on the periodic times; and in analysis motions in one direction have a different sign from those in a contrary direction: but as all the planets and satellites revolve from west to east, the radicals, and consequently all the terms of the preceding equations must have positive signs; therefore each term is less than the constant quantity C.

But observation shows that the orbits of the planets and satellites are nearly circular, hence each of the quantities

$$e^2 m\sqrt{a},\ e'^2 m'\sqrt{a'},\ \&\text{c.}$$

is very small; and C being a very small constant quantity given by observation, the first number of equation (136) is very small.

As C never could have changed since the system was constituted as it now is, so it never can change while the system remains the same; therefore equation (136) cannot contain any quantity that increases indefinitely with the time; so that none of the roots g, g_1, g_2, &c., are either equal or imaginary.

492. Since the greater axes and masses are invariable, and the eccentricities are perpetually changing, they have the singular property of compensating each other's variation, so that the sum of their squares, respectively multiplied by the coefficients $m\sqrt{a}$, $m'\sqrt{a'}$, &c., remains constant and very small.

493. To remove all doubts on a point so important, suppose some of the roots, g, g_1, g_2, &c., to be imaginary, then some of the cosines or sines will be changed into exponentials; and, by article 215, the general value of h in (133) would contain the term Cc^{at}, c being the number whose hyperbolic logarithm is unity. If Dc^{at}, $C'c^{at}$, $D'c^{at}$, &c., be the corresponding terms introduced by these imaginary roots in h, h', l', &c., the e^2 would contain a term $(C^2 + D^2)c^{2at}$, e'^2 would contain $(C'^2 + D'^2)c^{2at}$, and so on; hence the first number of equation (136) would contain [30]

$$c^{2at}\left\{m\sqrt{a}\left(C^2 + D^2\right) + m'\sqrt{a'}\left(C'^2 + D'^2\right) + \&\text{c.}\right\},$$

a quantity that increases indefinitely with the time.

If c^{at} be the greatest exponential that in h, l, h', l', &c., contain, c^{2at} will be the greatest in the first member of equation (136); therefore the preceding term cannot be destroyed by any other term in that equation. In order, therefore, that its first member may be reduced to a constant quantity, the coefficient of c^{2at} must itself be zero; hence

$$m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)+\&c.=0\,.$$

But if the radicals $\sqrt{a}$, $\sqrt{a'}$, &c., have the same sign, that is, if all the bodies m, m', &c., move in the same direction, this coefficient can only be zero when each of the quantities C, D, C', D', &c., is zero separately; thus, h, l, h', l', &c., do not contain exponentials, and therefore the roots of [31] g, g_1, &c., are all real. If the roots g and g_1 be equal, then the preceding integral becomes[32]

$$h=\left(b+b'\right)c^{at}=\left(b+b'\right)\left(1+\frac{at}{2}+\frac{a^2t^2}{1.2}+\&c.\right).$$

Thus the general value of h will contain a finite number of terms of the form Ct^r, which increases indefinitely with the time; the same roots would introduce the terms Dt^r, $C't^r$, $D't^r$, &c., in the general value of l, h', l', &c.; therefore the first member of equation (136) would contain the term

$$t^2\left\{m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)\right\}+\&c.=0\,;$$

and if t^r be the highest power of t in h, l, h', l', &c.; t^{2r} will be the highest power of t in equation (136); consequently its first member can only be constant when

$$m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)+\&c.=0\,,$$

which cannot happen when all the planets revolve in the same direction, unless

$$C=0,\ D=0,\ C'=0,\ D'=0,\ \&c.$$

Thus, h, l, h', l', &c., neither contain exponentials nor circular arcs, when the bodies of the solar system revolve in the same direction, and as they really do so, the roots g, g_1, g_2, &c., are all real and unequal.

494. Because the equation (135) does not contain any quantity that increases with the time, on account of the roots[33] g, g_1, &c., being real and unequal, and that the eccentricities themselves and their variations are extremely small, the eccentricities increase and decrease with the cosines, between fixed but very narrow limits, in immense periods: for, considering only the mutual disturbances of Jupiter and Saturn, the eccentricities of their orbits would take no less than 70,414 years to accomplish their changes; but if more than two planets be taken, and compound periods established, they would evidently extend to millions of years.

495. The positions of the perihelia now remain to be considered.

$$e\sin v = h,\ e\cos v = l \text{ give } \tan v = \frac{h}{l},$$

and substituting the values of h and l in article 483,[34]

$$\tan v = \frac{N\sin(gt + x) + N_1 \sin(g_1 t + x_1) + \&\text{c.}}{N\cos(gt + x) + N_1 \cos(g_1 t + x_1) + \&\text{c.}};$$

or, if $gt + x$ be subtracted from v,

$$\tan(v - gt - x) = \frac{\tan v - \tan(gt + x)}{1 + \tan v\ \tan(gt + x)};$$

and when substitution is made for $\tan v$,

$$\tan(v - gt - x) = \frac{N'\sin\{(g_1 - g)t + x_1 - x\} + N_2 \sin\{(g_2 - g)t + x_2 - x\}}{N + N_1 \cos\{(g_1 - g)t + x_1 - x\} + N_2 \cos\{(g_2 - g)t + x_2 - x\} + \&\text{c.}}.$$

This tangent never can be infinite, if the sum $N + N_1 + N_2 + \&\text{c.}$, of the coefficients in the denominator be less than N with a positive sign; for in this case the denominator never can be zero; so that the angle $v - gt - x$ never can attain to a quadrant, but will oscillate between $+90^\circ$ and -90°; hence the true motion of the perihelion is $gt + x$.

From this equation it appears that the motions of the perihelia are not uniform, and that they may experience variations in the course of ages, to which no limits can be assigned, though observation shows that the variations are very slow.

496. Because the equations which give the secular variations in the eccentricities and longitudes of the perihelia do not contain the mean longitudes nor the inclinations of the orbits, they are independent of the configuration of the planets, and would be the same if all the bodies revolved in one plane, at least when the approximation does not extend to the higher powers of the eccentricities, inclinations, or masses. These secular inequalities depend on the angular distances of the perihelia of all the planets taken two and two, that is, on the configuration of the orbits.

497. It may be concluded from the preceding analysis, that when periodic inequalities are omitted, the mean motions of the planets are uniform; and that the system is stable with regard to the species of the orbits, which, retaining the greater axis invariable, deviate but little from the circular form; the eccentricities being subject to the condition expressed by equations (136)–that the sum of their squares, multiplied by the masses of the bodies, and the square roots of the greater axes of their orbits is invariably the same. The perihelia alone are subject to unlimited variations.

Secular Variations in the Inclinations of the Orbits and Longitudes of their Nodes

498. In order to determine the secular inequalities in the inclinations of the orbits and longitudes of the nodes, let the equations in article 474 be resumed

$$\frac{dp}{dt}=(0.1)(q'-q)$$

and

$$\frac{dq}{dt}=-(0.1)(p'-p),$$

which express the variations in the position of the orbit of m, when troubled by m' alone. But as all the bodies in the system act simultaneously on m, each of them will produce a variation in the inclination of its orbit, and in the longitude of its nodes, similar to those caused by the action of m'; hence

$$\frac{dp}{dt}=(0.1)(q'-q)+(0.2)(q''-q)+\&\text{c.}$$
$$\frac{dq}{dt}=-(0.1)(p'-p)-(0.2)(p''-p)-\&\text{c.}$$

will express the whole action of the system on the position of the orbit of m. Similar equations must exist for every body in the system: there will consequently be the following series of equations,

$$\begin{aligned}\frac{dp}{dt}&=-\{(0.1)+(0.2)+\&\text{c.}\}q+(0.1)q'+(0.2)q''+\&c.\\ \frac{dq}{dt}&=\{(0.1)+(0.2)+\&\text{c.}\}p-(0.1)p'-(0.2)p''-\&c.\\ \frac{dp'}{dt}&=-\{(0.1)+(1.2)+\&\text{c.}\}q'+(1.0)q+(1.2)q''+\&c.\\ \frac{dq'}{dt}&=\{(1.0)+(1.2)+\&\text{c.}\}p'-(1.0)p-(1.2)p''-\&c.\\ &\&\text{c.} \qquad \&\text{c.}\end{aligned} \qquad \textbf{(137)}$$

These equations are perfectly similar to those in article 481, and may be integrated on the same principle; whence

$$\begin{aligned}p&=N\sin(gt+x)+N_1\sin(g_1t+x_1)+\&\text{c.}\\ q&=N\cos(gt+x)+N_1\cos(g_1t+x_1)+\&\text{c.}\\ p'&=N'\sin(gt+x)+N_1'\sin(g_1t+x_1)+\&\text{c.}\\ q'&=N'\cos(gt+x)+N_1'\cos(g_1t+x_1)+\&\text{c.}\end{aligned} \qquad \textbf{(138)}$$

Stability of the Solar System with regard to the Inclination of the Orbits

499. The equation in g resulting from these, $g,\ g_1,\ g_2$, &c., for its roots, and the constant quantities $N,\ N_1$, &c. and $x,\ x_1$, &c. are determined. in a similar manner to what was employed for the eccentricities. For since $\bar{f},\ \bar{q}$, &c. are the values of $f,\ q$, &c. when $t=0$,[35]

$$\begin{array}{ll} p\ = \tan\bar{f}\sin\bar{q} & q = \tan\bar{f}\cos\bar{q}, \\ p' = \tan\bar{f}'\sin\bar{q}' & q' = \tan\bar{f}'\cos\bar{q}', \\ \&c. & \&c. \end{array}$$

hence, if all the inclinations of the orbits of the planets, and the longitudes of their nodes be known by observation at any given epoch, when $t=0$, there will be a sufficient number of equations to determine all the quantities $N,\ N_1$, &c. and $x,\ x_1$, &c.

500. Also the roots $g,\ g_1$, &c., are real and unequal, for if the equations (137) be respectively multiplied by[36]

$$m\sqrt{a}\,.\,p;\quad m\sqrt{a}\,.\,q;\quad m'\sqrt{a'}\,.\,p';\quad m'\sqrt{a'}\,.\,q';\ \&c.$$

and added, the integral of their sum will be

$$\left(p^2+q^2\right)m\sqrt{a}+\left(p'^2+q'^2\right)m'\sqrt{a'}+\&c.=C \qquad \textbf{(139)}$$

in consequence of the relations

$$\begin{array}{l} (0.1)\,m\sqrt{a}=(1.0)\,m'\sqrt{a'}, \\ (0.2)\,m\sqrt{a}=(2.0)\,m''\sqrt{a''}, \\ \quad\&c. \qquad \&c. \end{array}$$

Whence we may be assured by the same reasoning employed with regard to the eccentricities, that this equation neither contains arcs of circles nor exponentials, when the bodies all revolve in the same direction, so that all the roots are real and unequal.

501. Now $\tan f=\sqrt{p^2+q^2}$, and if the values of p and q be substituted[37]

$$\tan f=\sqrt{p^2+q^2}=$$
$$\sqrt{\left\{N^2+N_1^{\,2}+\&c.+2NN_1\cos\left\{(g_1-g)t+x_1-x\right\}+2NN_2\cos\left\{(g_2-g)t+x_2-x\right\}+\&c.\right\}}.$$

The expression $\sqrt{p^2+q^2}$ is less than $N+N_1+N_2+\&\text{c.},$ on account of these coefficients being multiplied by cosines which diminish their values. The maximum of $\tan f$ would be $N+N_1+\&\text{c.},$ which it never can attain, since the differences of the roots $g_1-g,\ g_2-g$ are never zero; and as the inclinations of the orbits of the planets on the plane of the ecliptic are very small, the coefficients $N,\ N_1,$ &c., which depend on the inclinations, are very small also, and will always remain so. And the inclinations of the orbits will oscillate between very narrow limits in periods depending on the roots $g,\ g_1,$ &c.

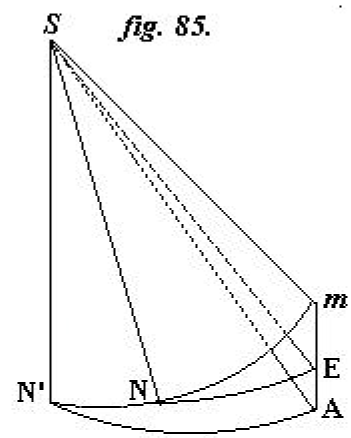


fig. 85.

502. The plane of the ecliptic in which the earth moves, changes its position in space from the action of the planets, each producing a retrograde motion in the intersection of the plane of the ecliptic, and that of its own orbit; whence it appears, that if EN be the orbit of the earth at a given epoch, AN' will be its position at a subsequent period, and so on. The secular inequality in the position or the terrestrial orbit changes the obliquity of the ecliptic; but as it is determined from equations (138) it oscillates between narrow limits, never exceeding 3°, therefore the equator never has coincided, and never will coincide with the ecliptic, supposing the system constituted as it is at present, so that there never was, and there never will be eternal spring.

503. Since $p^2+q^2=\tan^2 f,\ \ p'^2+q'^2=\tan^2 f',$ equation (139) becomes

$$m\sqrt{a}\tan^2 f+m'\sqrt{a'}\tan^2 f'+\&\text{c.}=C\,. \qquad \textbf{(140)}$$

Whence it may be concluded that the sum of the masses of all the bodies in the system multiplied by the square roots of half the greater axes of their orbits, and by the squares of the tangents of their inclinations on a fixed plane, will always be the same. If this sum be very small at any one period, and if all the radicals have the same sign, that is, if all the bodies revolve in the same direction, it will always remain so; and as in nature, the inclinations of all the orbits on the plane of the ecliptic are very small, and the bodies revolve in the same direction, the variations of the inclinations compensate each other, so that this expression will remain for ever constant, and very small.

504. Other two integrals may be obtained from the equations (137). For if the first be multiplied by $m\sqrt{a}$, the third by $m'\sqrt{a'}$, the fifth by $m''\sqrt{a''}$, &c., &c. their sum will be

$$m\sqrt{a}\,\frac{dp}{dt}+m'\sqrt{a'}\,\frac{dp'}{dt}+m''\sqrt{a''}\,\frac{dp''}{dt}+\&\text{c.}=0\,,$$

in consequence of the relations in article 484, the integral of which is

$$m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'+m''\sqrt{a''}\,.\,p''+\&\text{c.}=\text{constant}\,.$$

In a similar manner the differential equations in $q,\ q',$ give

$$m\sqrt{a}\,.\,q + m'\sqrt{a'}\,.\,q' + m''\sqrt{a''}\,.\,q'' + \&\text{c.} = \text{constant}\,.$$

505. With regard to the nodes $\tan q = \frac{p}{q}$, and substituting for *p* and *q*,

$$\tan q = \frac{N\sin(gt+x) + N_1\sin(g_1t+x_1) + \&\text{c.}}{N\cos(gt+x) + N_1\cos(g_1t+x_1) + \&\text{c.}};$$

or subtracting $gt+x$ from q,

$$\tan(q-gt-x) = \frac{N_1\sin\{(g_1-g)t+x_1-x\} + N_2\sin\{(g_2-g)t+x_2-x\} + \&\text{c.}}{N + N_1\cos\{(g_1-g)t+x_1-x\} + N_2\cos\{(g_2-g)t+x_2-x\} + \&\text{c.}}$$

If the sum of the coefficients $N+N_1+N_2+\&\text{c.}$ of the cosines in the denominator taken positively be less than *N*, $\tan(q-gt-x)$ never can be infinite; hence the angle $q-gt-x$ will oscillate between $+90^\circ$ and -90°, so that $gt+x$ is the true motion of the nodes of the orbit of *m*, and $g = \frac{360^\circ}{\text{period of } \Omega \text{ of } m}$. As in general the periods of the motions of the nodes are great, the inequalities increase very slowly. From these equations it may be seen, that the motion of the nodes is indefinite and variable.

The method of computing the constant quantities will be given in the theory of Jupiter, whence the laws, periods, and limits of the secular variations in the elements of his orbit, will be determined.

506. The equations which give[38] $p,\ q,\ p',\ \in$ may be expressed by a diagram. Let A*n* be the orbit of the planet *m* at any assigned time, as the beginning of January, 1750, which is the epoch of many of the French tables. After a certain time, the action of the disturbing body m' alone on the planet *m*, changes the inclination of its orbit, and brings it to the position B*n*. But m'' acting simultaneously with m' brings the orbit into the position *Cn*: m'' acting along with the preceding bodies changes it to Dn'', and so on. It is evident that the last orbit will be that in which *m* moves. So the whole inclination of the orbit of *m* on the plane A*n*, after a certain time, will be the sum of the finite and simultaneous changes. Hence if N be the inclination of the circle B*n* on the fixed plane A*n*, and $g\,Sn = gt+x$ the longitude of its ascending node; N′ the inclination of the circle Cn' on B*n*, and

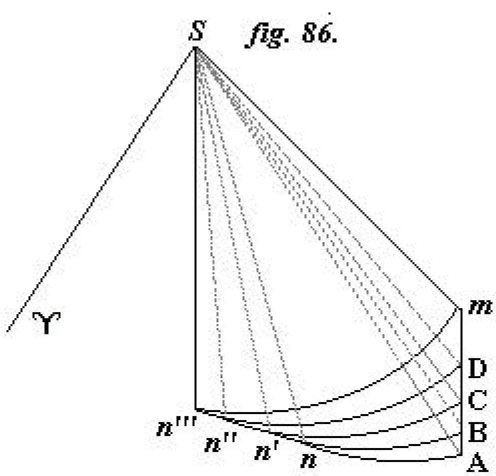


fig. 86.

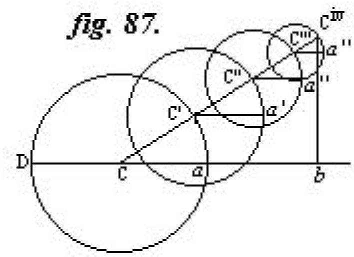
$g\,Sn' = g't + x'$ the longitude of the node n'; N'' the inclination of the circle $\mathrm{D}n''$ on $\mathrm{C}n'$, and $g\,Sn'' = g_2 t + x_2$ the longitude of the node n''; and so on for each disturbing body, the last circle will be the orbit of m.

507. Applying the same construction to h and l (133), it will be found that the tangent of the inclination of the last circle on the fixed plane is equal to the eccentricity of the orbit of m; and that the longitude of the intersection of this circle with the same plane is equal to that of the perihelion of the orbit of m.

508. The values of p and q in equations (138) may be determined by another construction; for let C, fig. 87,[39] be the centre of a circle whose radius is N; draw any diameter Da, and take the arc

$$a\mathrm{C}' = gt + x\,;$$

fig. 87.



on C' as a centre with radius equal to[40] N_1, describe a circle, and having drawn $\mathrm{C}'a'$ parallel to Ca, take[41] $a'\mathrm{C}' = g_1 t + x_1$: on C'' as it centre with radius equal to N_2, describe a circle, and having drawn $\mathrm{C}''a''$ parallel to Ca, take the $a''\mathrm{C}'' = g_2 t + x_2$, and so on. Let $a^{\text{iv}}C^{\text{iv}}$ be the arc in the last circle, then if $\mathrm{C}^{\text{iv}}b$ be perpendicular to Ca produced, it is evident that

$$\mathrm{C}^{\text{iv}}b = p, \quad \mathrm{C}b = q,$$

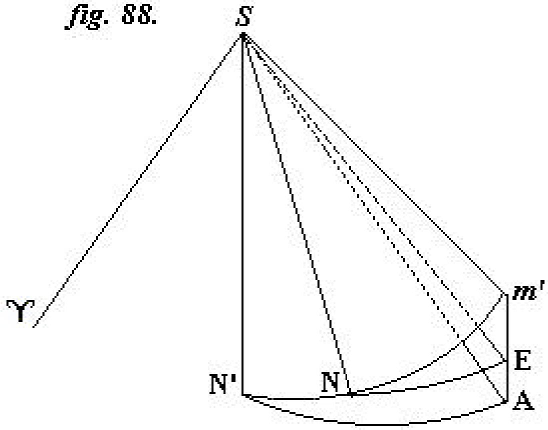
and if CC^{iv} be joined,

$$\tan f = \sqrt{p^2 + q^2}, \quad \tan q = \frac{p}{q},$$

q being the angle $\mathrm{C}^{\text{iv}}\mathrm{C}b$.

509. The equations which determine the secular variations in the inclinations and motions of the nodes being independent of the eccentricities, are the same as if the orbits of the planets were circular.

fig. 88.



Annual and Sidereal Variations in the Elements of the Orbits, with regard to the variable Plane of the Ecliptic

510. Equations (128) give the annual variations in the inclinations and longitudes of the nodes with regard to a fixed plane, but astronomers refer the celestial motions to the movable orbit of the earth whence observations are made; its motion occasioned by the action of the planets is indeed extremely minute, but it is important to know the secular variations in the position of the orbits with regard to it. Suppose AN fig. 88, to be the plane of the ecliptic or orbit of the earth, EN the variable plane of the ecliptic in which the earth is moving at a subsequent period, and $m'\mathrm{N}'$ the orbit of a planet m', whose position with regard to EN is to be determined.

By article 444,

$$\mathrm{EA} = q\sin\left(n't+\in'\right) - p\cos\left(n't+\in'\right)$$

is the latitude of *m* above AN; and the latitude of m' above AN' is

$$\mathrm{A}m' = q'\sin\left(n't+\in'\right) - p'\cos\left(n't+\in'\right).$$

As the inclinations are supposed to be very small, the difference of these two, or $m'\mathrm{A}-\mathrm{EA}$ is very nearly equal to $m'\mathrm{E}$ the latitude of m' above the variable plane of the ecliptic EN.

If $\grave{f}$ be the inclination of $m'\mathrm{N}'$ the orbit of m' to EN the variable ecliptic, and $\grave{q}$ the longitude of its ascending node, then will

$$\tan\grave{f}.\sin\grave{q} = p'-p; \quad \tan\grave{f}.\cos\grave{q} = q'-q.$$

Whence[42]

$$\tan\grave{f} = \sqrt{\left(p'-p\right)^2+\left(q'-q\right)^2}; \ \tan\grave{q} = \frac{p'-p}{q'-q}.$$

If EN be assumed to be the fixed plane at a given epoch, then $p=0$, $q=0$, but neither *dp* nor *dq* are zero; hence[43]

$$d\grave{f} = \left(dp'-dp\right).\sin q' + \left(dq'-dq\right).\cos q',$$
$$d\grave{q} = \frac{\left(dp'-dp\right).\cos q' - \left(dq'-dq\right).\sin q'}{\tan f'},$$

and substituting the values in article 498 in place of the differentials *dp*, *dq*, &c. there will result

$$\frac{d\grave{f}}{dt} = \left\{(1.2)-(0.2)\right\}\tan f''\sin\left(q'-q''\right) + \left\{(1.3)-(0.3)\right\}\times\tan f'''\sin\left(q'-q'''\right) + \&c.$$

$$\begin{aligned}\frac{d\grave{q}}{dt} = &-\left\{(1.0)+(1.2)+(1.3)+\&c.\right\}-(0.1)\\ &+\left\{(1.2)-(0.2)\right\}\cdot\frac{\tan f''}{\tan f'}\cdot\cos\left(q'-q''\right)\\ &+\left\{(1.3)-(0.3)\right\}\frac{\tan f'''}{\tan f'}\cos\left(q'-q'''\right)\\ &+\&c.\end{aligned} \qquad \textbf{(141)}$$

Motion of the Orbits of two Planets

511. Imagine two planets m and m' revolving round the sun so remotely from the rest of the system, that they are not sensibly disturbed by the other bodies.

Let $g=\sqrt{(p'-p)^2+(q'-q)^2}$ be the mutual inclination of the two orbits supposed to be very small. If the orbit of m at the epoch be assumed as the fixed plane

$$f=0,\ \ g=f',\ \ p=0,\ \ q=0,$$

and

$$\tan^2 f'=\tan^2 g=p'^2+q'^2 .$$

In this case, equations (140) and (128) become

$$m'\sqrt{a'}\tan^2 f'=C,\quad \frac{dq'}{dt}=-(0.1) .$$

Since the greater axes of the orbits are constant, the first shows that the inclination is constant, and the second proves the motion of the node of the orbit of m' on that of m to be uniform and retrograde, and the motion of the intersection of the two orbits on the orbit of m, in consequence of their mutual attraction, will be $-(0.1)t$.

Secular Variations in the Longitude of the Epoch

512. The mean place of a planet in its orbit at a given instant, assumed to be the origin of the time, is the longitude of the epoch. It is one of the most important elements of the planetary orbits, being the origin whence the antecedent and subsequent longitudes are estimated. If the mean place of the planet at the origin of the time should vary from the action of the disturbing forces, the longitudes estimated from that point would be affected by it; to ascertain the secular inequalities of that element is therefore of the greatest consequence.

The differential equation of the longitude of the epoch in article 441, is

$$d\in=\frac{an\sqrt{1-e^2}}{e}\cdot\left(1-\sqrt{1-e^2}\right)\cdot\frac{dF}{de}dt-2a^2n\frac{dF}{da}dt .$$

By article 473,

$$\frac{dF}{da}=-\frac{m'}{2}\cdot\left(\frac{3a'S'+2aS}{\left(a'^2-a^2\right)^2}\right)$$

$$-\frac{m'}{4}\cdot\frac{a'^2}{a}\cdot\left(\frac{3S'\left(2a'^2-3a^2\right)+6aa'S}{\left(a'^2-a^2\right)^3}\right)ee'\cos\left(v'-v\right)$$

$$+\frac{m'}{2.4}aa'\left(\frac{6a'S-3aS'}{\left(a'^2-a^2\right)^3}\right)\left\{e^2+e'^2-\left(p'-p\right)^2-\left(q'-q\right)^2\right\}$$

$$\frac{dF}{de}=-\frac{3m'aa'.S'}{4\left(a'^2-a^2\right)^2}\cdot e$$

$$+\frac{3m'}{2\left(a'^2-a^2\right)^2}\left\{\left(a'^2+a^2\right)S'+aa'S\right\}e'\cos\left(v'-v\right).$$

If these be put in the value of $d\in$, rejecting the powers of e above the second, and if to abridge

$$C=\frac{m'.na^2.\left(2aS+3a'S'\right)}{\left(a'^2-a^2\right)^2},$$

$$C_1=-\frac{3m'.na^2a'\left(4aa'S-\left(3a^2-a'^2\right)S'\right)}{2.4.\left(a'^2-a^2\right)^3},$$

$$C_2=-\frac{3m'.na.\left\{\left(a^2-5a'^2\right)aa'S+\left(a^4+6a^2a'^2-5a'^4\right)S'\right\}}{4.\left(a'^2-a^2\right)^3},$$

$$C_3=\frac{3m'.na^3a'\left(2a'S-aS'\right)}{4\left(a'^2-a^2\right)^3},$$

$d\in$ becomes

$$\frac{d\in}{dt}=C+C_1e^2+C_2ee'\cos\left(v'-v\right)$$

$$+C_3\left\{\left(p'-p\right)^2+\left(q'-q\right)^2-e'^2\right\}.$$

But

$$h=e\sin v \quad l=e\cos v,$$
$$h'=e'\sin v' \quad l'=e'\cos v';$$

hence

$$\frac{d\in}{dt}=C+C_1\left(h^2+l^2\right)+C_2\left(hh'+ll'\right)+C_3\left\{\left(p'-p\right)^2+\left(q'-q\right)^2-h'^2-l'^2\right\}.$$

513. This equation only expresses the variation in the epoch of m when troubled by m'; but, in order to have the effect of the whole system in disturbing the epoch of m, a similar set of terms must be added for each of the planets; but if the two planets m and m' alone be considered, their mutual inclination will be constant by article 511, hence $g^2 = (p' - p)^2 + (q' - q)^2 = M^2$, a constant quantity.

Again by article 483,[44]

$$h^2 + l^2 = N^2 + N_/^{\,2} + 2NN_/ \cos\{(g_/ - g)t + x_/ - x\}$$
$$h'^2 + l'^2 = N'^2 + N_/'^2 + 2N'N_/' \cos\{(g_/ - g)t + x_/ - x\}$$
$$hh' + ll' = NN' + N_/N_/' + (NN_/' + N'N_/)\cos\{(g_/ - g)t + x_/ - x\}.$$

Substituting these in $d\in$, and to abridge, making

$$A`n = C + C_/(N^2 + N_/^2) + C^2(NN' + N'N_/') + C_3(M^2 - N'^2 - N_/'^2),$$
$$B` = 2C_/NN' - 2C_3N'N_/' + C_2(NN_/' + N_/N'),$$

it becomes[45]

$$d\in = A`.ndt + B`\cos\{(g_/ - g)t + x_/ - x\}dt.$$

The integral of which is

$$d\in = A`nt + \frac{B`}{g_/ - g}\sin\{(g_/ - g)t + x_/ - x\}.$$

514. The term $A`nt$ only augments the mean primitive motion of the planet $m$ in the ratio of 1 to $1 + A'$, so that the mean motion which should result from observation would be $(1 + A')nt$, corresponding to the mean distance $\frac{a}{(1 + A`)^{\frac{2}{3}}}$.

Knowing this distance, which is given by a comparison of the periodic times, the primitive distance a may be determined; but as $A`$ is an infinitely small fraction of the order of the masses $m$ and $m'$, this correction in the mean distance is insensible. The term $A`nt$ may therefore be omitted, so that the secular variation in the epoch is[46]

$$d\in = \frac{B`}{g_/ - g}\sin\{(g_/ - g)t + x' - x\}. \qquad \textbf{(142)}$$

The variation in the epoch, like the other secular inequalities in article 480, may be expressed in series ascending according to the powers of the time; but as the term depending on its first power is insensible, it will have the form

$$d\in = Ht^2 + \&c.$$

This inequality is insensible for the planets; its greatest effect is produced in the theory of Jupiter and Saturn: but even then it is only $d\in = -0.0000006501''.t^2$ for Jupiter, and for Saturn $d\in' = +0''.0000015114.t^2$, t being any number of Julian years from 1750. This inequality is not the 60th part of a sexagesimal second in a century, a quantity altogether insensible. Like all other inequalities it is periodic; but its period, which depends on $g_{/}-g$, is for Jupiter and Saturn no less than 70,414 years. The variation $d\in$, though of the order of disturbing forces, may, in the course of many centuries, become sensible, on account of the small divisor $g_{/}-g$ introduced by integration; but although it is insensible with regard to the planets, it is of much importance in the theories of the Moon and of Jupiter's Satellites.

Stability of the System, whatever may be the powers of the Disturbing Masses

515. The stability of the system has been proved with regard to the greater axes of the orbits, even when the approximation extends to the squares of the disturbing forces, and to all powers of the eccentricities and inclinations. Its invariability with regard to the other elements has only been proved on the hypothesis of the orbits being nearly circular, and very little inclined to each other and to the plane of the ecliptic; but as the same results may be derived from the general equations of the motion of a system of bodies, they equally exist whatever the eccentricities and inclinations may be, and when the approximation includes the squares of the disturbing forces, and they remain the same whatever changes the secular inequalities may introduce in the lapse of ages.

516. If the equations of the motion of a system of bodies in article 346 be resumed, and the equations in $x,\ x'$, &c., multiplied respectively by

$$my-m\cdot\frac{\sum m\,.\,y}{S+\sum m};\ \ m'y'-m'\cdot\frac{\sum m\,.\,y}{S+\sum m};\ \ \text{\&c.}$$

and those in $y,\ y'$, &c. by

$$-mx+m\cdot\frac{\sum m\,.\,x}{S+\sum m};\ \ -m'x'+m'\cdot\frac{\sum m\,.\,x}{S+\sum m};\ \ \text{\&c.}$$

their sum will be

$$\sum m\cdot\left(\frac{xd^2y-yd^2x}{dt^2}\right)+\frac{\sum .my}{S+\sum m}\cdot\sum m\cdot\frac{d^2x}{dt^2}-\frac{\sum .mx}{S+\sum m}\cdot\sum m\cdot\frac{d^2y}{dt^2};$$

for the nature of the function l gives

$$y\cdot\frac{d\,l}{dx}+y'\cdot\frac{d\,l}{dx'}+\text{\&c.}=0;\ \ -x\cdot\frac{d\,l}{dy}-x'\cdot\frac{d\,l}{dy'}-\text{\&c.}=0\,,$$
$$\frac{d\,l}{dx}+\frac{d\,l}{dx'}+\text{\&c.}=0;\ \ \frac{d\,l}{dy}+\frac{d\,l}{dy'}+\text{\&c.}=0;$$

as may be seen by trial. The integral of the preceding equation is

$$\sum m\cdot\left(\frac{xdy-ydx}{dt}\right)+\frac{\sum .my}{S+\sum m}\cdot\sum m\cdot\frac{dx}{dt}-\frac{\sum .mx}{S+\sum m}\cdot\sum m\cdot\frac{dy}{dt}=C\,.$$

A similar equation may be found in *x*, *z*, and *y*, z; and when $S+m'=1$, it will be found that[47]

$$\begin{aligned}
&\sum m\cdot\frac{ydx-xdy}{dt}+\sum mm'\left(\frac{xdy'-y'dx+x'dy-ydx'}{dt}\right)=C\\
&\sum m\cdot\frac{xdz-zdx}{dt}+\sum mm'\left(\frac{zdx'-x'dz+z'dx-xdz'}{dt}\right)=C'\\
&\sum m\cdot\frac{zdy-ydz}{dt}+\sum mm'\left(\frac{y'dz-zdy'+y'dz-zdy'}{dt}\right)=C''
\end{aligned}\qquad\textbf{(143)}$$

$C,\ C',\ C''$, being constant quantities. Now $\frac{ydx-xdy}{dt}$ is double the area described in the time *dt* by the projection of the radius vector of m on the plane xy. This area on the orbit is $\sqrt{a\left(1-e^2\right)}$; and if f be the inclination of the orbit on the plane xy, $\cos f\sqrt{a\left(1-e^2\right)}$ is its projection. In the same manner

$$\frac{y'dx'-x'dy'}{dt}=\cos f'\sqrt{a'\left(1-e'^2\right)}$$

is the area described by the projection of the radius vector of m' on the same plane, and so on. In consequence of these the first of the preceding equations becomes

$$m\sqrt{a\left(1-e^2\right)}\cos f+m'\sqrt{a'\left(1-e'^2\right)}\cos f'+\&\text{c.}=mm'\left(\frac{ydx'-x'dy+y'dx-xdy'}{dt}\right)+\&\text{c.}+C\,.$$

If the elliptical values of $x,\ y,\ x',\ y'$, be substituted, the first term of the second member of this equation must always be periodic; for, in consequence of the observations in article 466, the arcs $nt,\ n't$, never destroy one another in the expressions $ydx',\ x'dy$, &c. Hence, if periodic quantities and those of the fourth be neglected, the last number of the equation is constant. If the products $ydx',\ x'dy$, &c., contained constant terms, they would be of the first order with regard to the masses; and as they are functions of the elliptical elements, their variation is of the second order; consequently, the variation of the terms $mm'.\,y'dx$, &c. is of the fourth order. If the periodic part of the values of the elliptical elements be substituted in the first member of the preceding equation, any terms resulting from that substitution that are not periodic will be of the third order, and may be regarded as constant. The second member of the equation in question may therefore be esteemed constant. Hence,

$$m\sqrt{a\left(1-e^2\right)}\cos f+m'\sqrt{a'\left(1-e'^2\right)}\cos f'+\&\text{c.}=\text{C}\qquad\textbf{(144)}$$

517. Again, $\frac{xdz - zdx}{dt}$ and $\frac{zdy - ydz}{dt}$ are the areas described by the radius vector of m in the time dt, projected on the co-ordinate planes xz, and yz. But it is easy to see by trigonometry that the cosines of the inclination of the orbit on these planes are $\sin f \cos q$, and $\sin f \sin q$; hence

$$\frac{xdz - zdx}{dt} = \sqrt{a\left(1-e^2\right)} \sin f \cos q \ ,$$

and

$$\frac{zdy - ydz}{dt} = \sqrt{a\left(1-e^2\right)} \sin f \sin q \ .$$

Similar expressions exist for all the bodies; and as the same reasoning applies to the two last equations (143), as to the first, they give[48]

$$\begin{aligned} &m\sqrt{a\left(1-e^2\right)} \sin f \cos q + m'\sqrt{a'\left(1-e'^2\right)} \sin f' \cos q' + \&\text{c.} = C', \\ &m\sqrt{a\left(1-e^2\right)} \sin f \sin q + m'\sqrt{a'\left(1-e'^2\right)} \sin f' \sin q' + \&\text{c.} = C''. \end{aligned} \qquad \textbf{(145)}$$

518. These relations exist whatever the eccentricities and inclinations may be, and whatever may be the changes that they undergo in the course of ages from their secular inequalities, the approximation extending to the third order inclusively, and even to the squares of the disturbing forces.

519. A variety of results may be derived from them. Because

$$\cos f = \frac{1}{\sqrt{1+\tan^2 f}} \ ,$$

equation (144) gives

$$m\sqrt{\frac{a\left(1-e^2\right)}{1+\tan^2 f}} + m'\sqrt{\frac{a'\left(1-e'^2\right)}{1+\tan^2 f'}} + \&\text{c.} = C \ .$$

If e^4 and $e^2 f^2$ be omitted,

$$m\sqrt{\frac{a\left(1-e^2\right)}{1+\tan^2 f}} = m\sqrt{a\left(1-e^2\right)}\left(1+\tan^2 f\right)^{-\frac{1}{2}} = m\sqrt{a} - \tfrac{1}{2} m\sqrt{a}\left(e^2 + \tan^2 f\right),$$

consequently

$$\tfrac{1}{2} m\sqrt{a}\left(e^2 + \tan^2 f\right) + \tfrac{1}{2} m'\sqrt{a'}\left(e'^2 + \tan^2 f'\right) + \&\text{c.} = 2m\sqrt{a} + 2m'\sqrt{a'} + \&\text{c.} + 2C \ .$$

But the last member is altogether constant: hence

$$m\sqrt{a}\left(e^2+\tan^2 f\right)+m'\sqrt{a'}\left(e'^2+\tan^2 f'\right)+\text{\&c.} = \text{constant}.$$

It was shown that when the squares and products of the eccentricities and inclinations are omitted, the variations in the eccentricities are the same as if all the planets moved in one plane; and that the variations in the inclinations are the same as if the orbits were circular, as these quantities vary independently of one another, e, e', &c., and f, f', &c., may be alternately zero in the last equation, consequently,

$$m\sqrt{a}\,.\,e^2+m'\sqrt{a'}\,.\,e'^2+m''\sqrt{a''}\,.\,e''^2+\text{\&c.}=\text{constant};$$
$$m\sqrt{a}\,.\tan^2 f+m'\sqrt{a'}\,.\tan^2 f'+m''\sqrt{a''}\,.\tan^2 f''+\text{\&c.}=\text{constant};$$

results that are the same with equations (136) and (140).

If quantities of the order of the squares of the eccentricities and inclinations be omitted, the tangents of the very small quantities f, f', may be taken in place of their sines, so that by the substitution of [49]

$$\begin{aligned} p&=\tan f\sin q, & q&=\tan f\cos q,\\ p'&=\tan f'\sin q', & q'&=\tan f'\cos q',\\ &\text{\&c.} & &\text{\&c.}\end{aligned}$$

in equations (145) they become

$$m\sqrt{a}\,.\,q+m'\sqrt{a'}\,.\,q'+m''\sqrt{a''}\,.\,q''+\text{\&c.}=\text{constant},$$
$$m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'+m''\sqrt{a''}\,.\,p''+\text{\&c.}=\text{constant}.$$

520. Since the eccentricities and inclinations of all the orbits in the solar system are very small, the constant quantities in all the preceding equations of condition must be very small, provided the radicals $\sqrt{a}$, $\sqrt{a'}$, &c., have the same signs, that is, if the bodies all move in one direction, which is the case in nature; it may therefore be concluded that the elements vary within very narrow limits.

521. Let there be only two bodies m and m', the mutual inclination of their orbits being

$$\cos g=\cos f\cos f'+\sin f\sin f'\cos\left(q'-q\right);$$

then if the squares of the equations (144) and (145) be added, the result will be

$$m^2a\left(1-e^2\right)+m'a'\left(1-e'^2\right)+2mm'\sqrt{a\left(1-e^2\right)}\,.\sqrt{a'\left(1-e'^2\right)}\times\cos g=\text{constant}. \qquad \textbf{(146)}$$

Neglecting quantities of the fourth order, and putting all the constant quantities in the second member, it becomes

$$m\sqrt{a}\,.\,e^2 + m'\sqrt{a'}\,.\,e'^2 + \frac{4mm'\sqrt{aa'}\sin^2\tfrac{1}{2}g}{m\sqrt{a}+m'\sqrt{a'}} = \text{constant}\,,$$

for

$$\cos g = 1 - 2\sin^2\tfrac{1}{2}g\,.$$

The constant in the second part of this equation is equal to the first member at a given epoch, for at that epoch all the elements are supposed to be known by observation; it ought, therefore, to be independent of the variation of the elements $e,\ e'$, and g : its variation will be

$$m\sqrt{a}\,.\,ede + m'\sqrt{a'}\,.\,e'de' + \frac{2mm'\sqrt{aa'}\,.\,gdg}{m\sqrt{a}+m'\sqrt{a'}} = 0\,, \qquad \textbf{(147)}$$

for a and a' are constant. This relation must always exist among the secular variations of the eccentricities of the two orbits and their mutual inclination.

If the constant part of equation (146) be included in the second member it becomes

$$m^2ae^2 + m'^2a'e'^2 - 2mm'a^2a'^2nn'\sqrt{1-e^2}\sqrt{1-e'^2}\cos g = \text{constant}\,,$$

by the substitution of a^2n and a'^2n' for $\sqrt{a}$ and $\sqrt{a'}$; and if it be observed that

$$\sqrt{1-e^2} = 1 - \frac{e^2}{1+\sqrt{1-e^2}},\quad \sqrt{1-e'^2} = 1 - \frac{e'^2}{1+\sqrt{1-e'^2}},\quad \cos g = 1 - \frac{\sin^2 g}{1+\cos g}\,,$$

then will

$$\sqrt{1-e^2}\sqrt{1-e'^2}\cos g = 1 - \frac{e^2\sqrt{1-e'^2}\cos g}{1+\sqrt{1-e^2}} - \frac{e'^2\cos g}{1+\sqrt{1-e'^2}} - \frac{\sin^2 g}{1+\cos g}\,.$$

If this value be put in the preceding equation, and all constant quantities included in the second member, it becomes

$$m^2\,.\,ae^2 + m'^2\,.\,a'e'^2 + 2mm'\,.\,a^2a'^2nn'\cdot\frac{e^2\sqrt{1-e'^2}\,.\cos g}{1+\sqrt{1-e^2}}$$
$$+2mm'\,.\,a^2a'^2nn'\,.\frac{e'^2\cos g}{1+\sqrt{1-e^2}} + 2mm'\,.\,a^2a'^2nn'\,.\frac{\sin^2 g}{1+\cos g} = C_{/}\,;$$

$C_{/}$ being an arbitrary constant quantity.

$C_{/}$ is a very small quantity with regard to the squares and products of m and m', since they are multiplied by e^2, e'^2, $\sin g^2$; and that the mutual inclination of the two planes and their eccentricities are supposed to be very small, as is really the case in nature. Each term of the first member of this equation will therefore remain very small with regard to the squares and products of m and m'; if all the terms have the same sign, each term will then be less than $C_{/}$. But because all the planets revolve in the same direction round the sun, nt, $n't$, will have the same sign. Hence all the terms in the first member will be positive as long as g is less than 90°. But if $g = 90^\circ$, then $\sin g = 1$; $\cos g = 0$, which reduces the equation to

$$m^2 a e^2 + m'^2 a' e'^2 + 2mm' a^2 a'^2 nn' = C \,,$$

and the last term is no longer very small with regard to mm', which is impossible, since C_l is very small with regard to the product of m and m', and that the other terms of the first member are positive. Thus, because the angle g never can attain to 90°, it follows that g, the inclination, and the eccentricities e, e^2, of the two orbits, will always be small; for, as $\cos g$ never can become negative, every term in the first member of the equation under discussion will be positive, and will remain very small with regard to the squares and products of the masses m and m'. That is to say, the coefficients[50] e^2, e'^2, $\sin g^2$, will always remain very small, because they are small at present.

522. This reasoning would be the same whatever might be the number of planets, since each of them would only add terms to the first member of the equation under consideration, similar to those that compose it.

523. Thus it may be concluded that the planetary system is stable with regard to the eccentricities, the inclinations, and greater axes of the orbits, however far the approximation may be carried with regard to the elements of the orbits, even including the second powers of the disturbing forces.

524. Laplace[51] and Poisson[52] have proved the stability of the solar system when the approximation extends to the first and second powers of the disturbing force, on the hypothesis that all the planets revolve in nearly circular orbits, little inclined to each other; but in a very able paper read before the Royal Society on the 29th April, 1830, Mr. Lubbock[53] has shown that these conditions are not necessary in a system subject to the law of gravitation. He has obtained expressions for the variations of the elliptical constants, which are rigorously true, whatever the power of the disturbing force may be, whence it appears, that, however far the approximation may be carried, the eccentricities, the major axes, and the inclinations of the orbits to a fixed plane, contain no term that varies with the time, and that their secular variations oscillate between fixed limits in very long periods.

The Invariable Plane

525. It has been already mentioned that in the motion of a system of bodies there exists an invariable plane, which, always retaining a parallel position, is easily found, because the sum of the masses of the bodies of the system respectively multiplied by the projections of the areas described by their radii vectores in a given time, is a maximum on that plane, and the sum of the projections on any other planes at right angles to it is zero. It is principally in the solar system that this plane is of importance, on account of the proper motions of the stars, and of the plane of the ecliptic, which render it difficult to determine the celestial motions with precision, this difficulty indeed is already perceptible, and will increase when very accurate observations, separated by very long intervals of time, must be compared with each other.

If I be the inclination of the invariable plane on the fixed plane which contains the co-ordinates x and y, and if Ω be the longitude of its ascending node, by article 166

$$\tan I \sin\Omega = \frac{C''}{C}; \quad \tan I \cos\Omega = \frac{C'}{C};$$

and substituting the values of $C,\ C',\ C''$, given by equations (144) and (145),

$$\tan I \sin\Omega = \frac{m\sqrt{a\left(1-e^2\right)}\sin f \sin q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \sin q' + \&c.}{m\sqrt{a\left(1-e^2\right)}\cos f + m'\sqrt{a'\left(1-e'^2\right)}\cos f' + \&c.}$$

$$\tan I \cos\Omega = \frac{m\sqrt{a\left(1-e^2\right)}\sin f \cos q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \cos q' + \&c.}{m\sqrt{a\left(1-e^2\right)}\cos f + m'\sqrt{a'\left(1-e'^2\right)}\cos f' + \&c.}.$$

The second members of these two equations have been proved to be invariable, even in carrying out the approximation to the squares and products of the masses, whatever changes the secular variations may induce in the course of ages; and, by what Mr. Lubbock has shown, they must be constant, whatever the power of the disturbing force may be: hence it follows, that the invariable plane retains its position, notwithstanding the secular variations in the elliptical elements of the planetary system.

526. The determination of this plane requires a knowledge of the masses of all the bodies in the system, and of the elements of their orbits. Approximate values of these are only known with regard to the planets, but of the masses of the comets we are in total ignorance; however, as the mutual gravitation of the planets is sufficient to represent all their inequalities, it shows that, hitherto at least, the action of the comets on the planetary system is insensible. Besides, the comet of 1770 approached so near to the earth that its periodic time was increased by 2.046 days; and if its mass had been equal to that of the earth, it would have increased the length of the sidereal year by nearly one hour fifty-six minutes, according to the computation of Laplace; but he adds, that if an increase of only two seconds had taken place in the length of the year, it would have been detected by Delambre,[54] when he computed his astronomical tables from the observations of Dr. Maskelyne;[55] whence the mass of the comet must have been less than the $\frac{1}{3{,}000}$ part of the mass of the earth. The same comet passed through the satellites of Jupiter in the years 1767 and 1779, without producing the smallest effect. Thus, though comets are greatly disturbed by the action of the planets, they do not appear to produce any sensible effects by their reaction.

527. If the position of the ecliptic in the beginning of 1750 be assumed the fixed plane of the co-ordinates x and y, and if the line of the equinoxes be taken as the origin of the longitudes, it is found that at the epoch 1750 the longitude of the ascending node of the invariable plane was

$\Omega = 102^\circ\ 57'\ 30''$, and its inclination on the ecliptic $I = 1^\circ\ 35'\ 31''$; and if the values of the elements for 1950 be substituted in the preceding formulae, it will appear that in 1950

$$\Omega = 102^\circ\ 57'\ 15''; \ I = 1^\circ\ 35'\ 31'';$$

which differ but little from the first.

528. The position of this plane is really approximate, since it has been determined in the hypothesis of the solar system being an assemblage of dense points mutually acting on one another, whereas the celestial bodies are neither homogeneous nor spherical; but as the quantities omitted have hitherto been insensible, the position of the plane as it is here given, will enable future astronomers to ascertain the real changes that may have taken place in the forms and positions of the planetary orbits.

Notes

[1] This *reads* "contain" in the 1st edition (published erratum).

[2] The third term *reads* $\frac{d}{de} d\,e$ in the 1st edition (published erratum).

[3] The term $i'n't$ *reads* $i'nt$ in the 1st edition (published erratum).

[4] The term $i'n't$ *reads* $in't$ in the 1st edition (published erratum).

[5] In the 1st edition $\frac{1}{r^3} - \frac{1}{r'^3}$ *reads* $\frac{1}{r^3} - \frac{1}{r'^2}$.

[6] The last term in the 1st edition *reads* $\frac{dR'}{dq'} d\,q$.

[7] An error in a published erratum would replace $\frac{1}{r^3} - \frac{1}{r^3}$ with $\frac{1}{r^3} - \frac{1}{r'^2}$. The replacement should be $\frac{1}{r^3} - \frac{1}{r'^3}$ (see next note).

[8] A misplaced parenthesis *reads* $m'd'\left\{xx' + yy' + zz'\left(\frac{1}{r^3} - \frac{1}{r^3}\right)\right\}$ in the 1st edition.

[9] The prime in $-\frac{1}{r'^3}$ is misplaced and *reads* $-\frac{1}{r^{3\prime}}$ in the 1st edition.

[10] This *reads* r' *for* a' in the 1st edition (published erratum).

[11] This *reads* m' in the 1st edition (published erratum).

[12] This *reads* "elliptical co-ordinates of m.," in the 1st edition.

[13] See note 1, *Book1, Chapter 6*.

[14] These expressions *read* $\text{Sin}'\, i\left(n't - nt + \in' - \in\right) = 0$ and $\text{C}osi\left(n't - nt + \in' - \in\right) = 1$ in the 1st edition.

[15] The third differential *reads* $\frac{dp}{dt} = .\,an \cdot \frac{dF}{dq}$ in the 1st edition.

[16] The right hand side of the first expression *reads* (0,1) in the 1st edition.

[17] The 1st edition contains the element $\frac{\tan f'}{\tan f'}$ in the second equation. This should be $\frac{\tan f'}{\tan f}$ as in equation (128).

[18] This *reads* m', m'', &c. in the 1st edition.

[19] The second term *reads* $\frac{1}{2}\cdot\frac{d^3\overline{e}}{dt^2}t^2$ in the 1st edition (published erratum).

[20] The second term in the 2nd equation *reads* $-\boxed{0.1}\,h'$ in the 1st edition (published erratum).

[21] Punctuation added after first term.

[22] The subscript in g_1 is omitted in the 1st edition.

[23] The expression is printed with mismatched parentheses as $(a^2 - 2aa'\cos b + a'^2\}^{\frac{1}{2}}$ in the 1st edition.

[24] The 1st edition contains a missing multiplier and a misplaced comma written, $\boxed{0.1}\, m, n'a' = \boxed{1.0}\; .m'.na$.

[25] The 3rd term in the 2nd equation *reads* $N''N_2''m'\sqrt{a''}$ in the 1st edition.

[26] The first term in the numerator in the 1st edition contains an error, $\overline{e}$ *reads* *e*.

[27] This *reads* $g = \frac{360^0}{113270} = 19'\,4''.7$ in the 1st edition (published erratum).

[28] The 2nd, 3rd and 4th instances of the subscript 1 in N_1, g_1, and x_1 *read* as $N_/$, $g_/$, and $x_/$ in the 1st edition.

[29] As in equation (135) the three instances of the subscript 1 in N_1, *read* as $N_/$ in the 1st edition.

[30] Throughout this article the 1st edition text *reads* C^{at} and C^{2at} *for* c^{at} and c^{2at} .

[31] The next two instances of g_1 *read* $g_/$ in the 1st edition.

[32] The right hand parenthesis and punctuation are omitted in the 1st edition.

[33] Again, the subscript on g_1 *reads* $g_/$ in the 1st edition.

[34] See the previous note regarding the subscripts on N_1, g_1, and x_1 throughout the development in articles 495-500.

[35] In the 1st edition the sines and cosines are reversed (published erratum).

[36] The accent on q' in the fourth member of this series is omitted in the 1st edition.

[37] See note 33 above regarding the subscripts on N_1, g_1, and x_1.

[38] Punctuation added after third and fourth elements.

[39] This is labeled fig. 90 in the 1st edition (published erratum).

[40] The 1st edition text uses the radius $\mathrm{N}_/$ for N_1.

[41] The 1st edition text expresses this as $a'\mathrm{C}' = g_/t + x_/$.

[42] Punctuation added.

[43] We have added the multiplier symbol in the second term of both expressions; also, there is a misplaced comma in the denominator of the second expression after $\tan f`$ in the 1st edition.

[44] A parenthesis is omitted in the third expression of $\cos\{g_/ - g)t + x_/ - x\}$ in the 1st edition.

[45] An unbalanced parenthesis in the 1st edition *reads* $\cos\{(g_/ - g)\,t + x_/ - x)dt$.

[46] Punctuation in the 1st edition is contained inside the parentheses.

[47] The first term in the 1st edition *reads* $\Sigma m\cdot\frac{xdx - xdy}{dt}$ (published erratum).

[48] A printing error on the right hand side of the second expression *reads* $+ = \&c.C''$ in the 1st edition.

[49] The 4th expression *reads* $q = \tan f'\cos q'$ in the 1st edition.

[50] Punctuation added.

[51] See note 4, *Introduction.*

[52] See note 1, *Book I, Chapter 6.*

[53] Lubbock, John William, Sir, 1803-1865, *Account of the "Traite sur le flux et reflux de la mer," of Daniel Bernoulli; and a treatise on the attraction of ellipsoids*, London : C. Knight, 1830.
[54] See note 54, *Preliminary Dissertation.*
[55] Maskelyne, Nevil, 1732-1811, astronomer, born in London, England. Maskelyne produced the *British Mariner's Guide* (1763) and published the first volume of the *Nautical Almanac* in 1765 (see note 22, *Preliminary Dissertation*). His inventions include the prismatic micrometer. In 1774 he measured the earth's density using a plumb line. His measurements showed that the density was 4.5 times that of water. He was also the first to make measurements of time with a precision to one tenth of a second.