BOOK II

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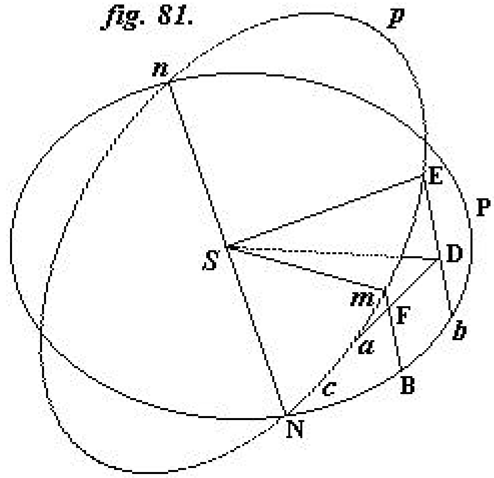
CHAPTER V

THEORY OF THE PERTURBATIONS OF THE PLANETS

410. THE tables computed on the theory of perfectly elliptical motion, are soon found inadequate to give the true place of a planet, on account of the reciprocal disturbances of the system. It is therefore necessary to investigate what these disturbances are, and to determine their effects.

In the first approximation to the celestial motions, the mutual action of the sun and one planet was considered: it then appeared that a planet, *m*, moves round the sun in an ellipse N*m*P*n*, fig. 81, inclined to the ecliptic NB*n*, at a very small angle P*np*. Now, if *m* be attracted by another planet m', which is much smaller than the sun, it will no longer go on in its elliptical orbit N*mn*, but will be drawn out of that orbit, and will move in some curved line, *ca*D, which may either be nearer to, or farther from, the plane of the ecliptic, according to the position of the disturbing body. In the first infinitesimal of time, the troubled orbit coincides with the ellipse through an indefinitely small space *ca*; in the second infinitely small interval of time, *am* will be the path of the planet in the ellipse, and *a*D will be its path in its troubled orbit: *am* is described in consequence of the action of the sun alone; *a*D by the combined action of the sun and of the disturbing body; *am* is the second increment of the space; *a*D is the second increment of the space, together with some very small space, FD, introduced by the action of the disturbing force. In consequence of the addition of FD, the longitude of *m* is increased by B*b*; its latitude is changed by the angle DSE, and the radius vector is increased by the difference between *S*D and *Sm*,–these three quantities are the perturbations of the planet in longitude, latitude, and distances.

fig. 81.



411. It is evident that the perturbations are true variations; and as the longitude, latitude, and radius vector of a planet moving in an elliptical orbit, have been represented by *v*, *s*, and *r*, the arcs $\text{B}b = dv$, $\text{ED} = ds$, and *S*D-$Sm - dr$, are the variations of these inequalities.

412. The perturbations in longitude, latitude, and distance, depend on the configuration of the bodies; that is, on the position of the bodies with regard to each other, to their perihelia and to their nodes. These inequalities, after going through a certain course of increase and decrease, are renewed as often as the bodies return to the same relative positions, and are therefore called Periodic Inequalities.

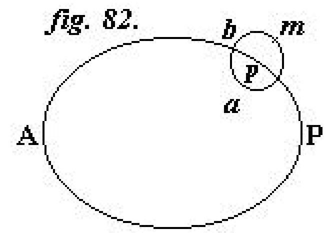
413. Thus the place of a planet, *m*, moving in its troubled orbit *ca*D, will be determined by the co-ordinates $v+dv$, $s+ds$, $r+dr$. These, however, are modified by a variation in the elements of the ellipse; for it is evident that, the path of the planet being changed from *a*E to *a*D, the elements of the ellipse N*m*E must vary. The variations of the elements are independent of the configuration or relative position of the bodies, and are only sensible in many revolutions; whereas those depending on the configuration, accomplish their changes in short periods. Thus $v+dv$, $s+ds$, $r+dr$, may be regarded as the co-ordinates of the planet in its true orbit, provided the elements contained in these functions be considered to vary by very slow degrees. This perfectly accords with observation, whence it appears that the perihelia of the orbits of the planets and satellites have a very slow direct motion in space; that the nodes have a slow retrograde motion; and that the eccentricities and inclinations are perpetually varying by very slow degrees. These very slow changes are really periodic, but many ages elapse before they accomplish their revolutions; on that account they are called Secular Inequalities, to distinguish them from the Periodic Inequalities, which pass rapidly from their maxima to their minima. Thus the Periodic Inequalities only depend on the configuration of the bodies, whereas the Secular Inequalities depend on the configuration of the perihelia and nodes alone.

414. Lagrange[1] took a new and very elegant view of the subject:–he considered the changes dv, ds, dr, to arise entirely from periodic and secular variations in the elements of elliptical motion, thus referring all the inequalities, to which a planet is liable, to changes in the elements of its orbit alone. In fact, as the curve *a*D very nearly coincides with the ellipse, it may be regarded as a portion of a new ellipse, having elements differing from those of the original one by infinitely minute variations. Of *these* a portion will be compensated in a whole revolution, or many revolutions of *m*, and of the disturbing planet constituting the Periodic Inequalities; but a portion will remain uncompensated, and entire1y independent of the position of the bodies with regard to each other. These uncompensated parts increase and diminish with extreme slowness; their effects on the motion of *m* partake of that character, and constitute what are called Secular Inequalities. Thus, in Lagrange's view, the co-ordinates of *m* in its elliptical orbit are modified, both by periodic and secular variations, in the elements of the ellipse.

415. The secular inequalities depend on the ratio of the disturbing mass to that of the sun, which, by article 350, is a very small fraction. Their arguments are not only different from those of the periodic inequalities, but, though also periodic, their periods are immensely longer.

416. Both periodic and secular inequalities may be represented by supposing a point *p* to revolve in an ellipse AP, fig. 82, where all the elements are perpetually varying by very slow degrees. Then, suppose a planet *m* to oscillate round the moveable point *p* in a curve *mab*, whose nature depends on the disturbing forces: this oscillating motion will represent the periodic inequalities, and the whole compound motion *m* represents the real motion of a planet in its troubled orbit.

fig. 82.



Demonstration of Lagrange's Theorem

417. The equations which determine the real motion of *m* in its troubled orbit are, by article 347,

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=\left(\frac{dR}{dx}\right),$$
$$\frac{d^2y}{dt^2}+\frac{my}{r^3}=\left(\frac{dR}{dy}\right), \qquad \textbf{(87)}$$
$$\frac{d^2z}{dt^2}+\frac{mz}{r^3}=\left(\frac{dR}{dz}\right).$$

If $R=0,$ these equations would be the same with those in article 365, already integrated. Let *a* be one of the arbitrary constant quantities, or elements of the orbit of *m*, introduced by integration. When $R=0,$ then[2]

$$a=\text{Func.}\left(x,y,z,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt},t\right)$$

may represent any one whatever of the integrals (91); or, if to abridge

$$x_{/}=\frac{dx}{dt}\quad y_{/}=\frac{dy}{dt}\quad z_{/}=\frac{dz}{dt},$$

[then]

$$a=\text{Func.}\left(x,y,z,x_{/},y_{/},z_{/},t\right). \qquad \textbf{(103)}$$

During the instant *dt*, the ellipse and troubled orbit coincide; therefore *x*, *y*, *z*, $x_{/}$, $y_{/}$, $z_{/}$ have the same values in both, and *a* is constant. But at the end of the instant *dt*, the velocities $x_{/}$, $y_{/}$, $z_{/}$, are respectively augmented, from the action of the disturbing forces, by the indefinitely small quantities[3]

$$\frac{dR}{dx}dt,\quad \frac{dR}{dy}dt,\quad \frac{dR}{dz}dt;$$

then *a* is no longer constant; and when $x_{/}$, $y_{/}$, $z_{/}$ are increased by those quantities, the corresponding variation of *a* is

$$da=\left(\frac{da}{dx_{/}}\cdot\frac{dR}{dx}+\frac{da}{dy_{/}}\cdot\frac{dR}{dy}+\frac{da}{dz_{/}}\cdot\frac{dR}{dz}\right)dt. \qquad \textbf{(104)}$$

If equation (103) be regarded as the first integral of the equations (87), when $R=0,$ it will evidently satisfy the same equations when *R* is not zero, because the values of *x*, *y*, *z*, $x_{/}dt$, $y_{/}dt$, $z_{/}dt$, are supposed to be the same in each orbit, since these quantities only differ in the two curves by their second differentials.

Hence, if $(x_/)$, $(y_/)$, $(z_/)$ be the values of $x_/$, $y_/$, $z_/$, when $R=0$, then

$$x_/=(x_/),\quad y_/=(y_/),\quad z_/=(z_/)\,,$$

and

$$dx_/=(dx_/)+\delta x_/,\quad dy_/=(dy_/)+\delta y_/,\quad dz_/=(dz_/)+\delta z_/\,.$$

Let func.$\left(x,y,z,x_/,y_/,z_/,t\right)$ be the differential of equation (103) when $R=0$, then will

$$0=\text{func.}\left(x,y,z,x_/,y_/,z_/,t\right)$$

and the differential of the same equation, when R is not zero, will be

$$da=\text{func.}\left(x,y,z,x_/,y_/,z_/,t\right)+\left(\frac{da}{dx_/}\delta x_/+\frac{da}{dy_/}\delta y_/+\frac{da}{dz_/}\delta z_/\right),$$

because, in the latter case, all the quantities vary. If the first differential be subtracted from the second, the result will be

$$da=\left(\frac{da}{dx_/}\delta x_/+\frac{da}{dy_/}\delta y_/+\frac{da}{dz_/}\delta z_/\right). \qquad \textbf{(105)}$$

But if

$$(dx_/)+\delta x_/,\quad (dy_/)+\delta y_/,\quad (dz_/)+\delta z_/\,,$$

be put, in equations (87), in place of their equals,

$$\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \frac{d^2z}{dt^2}\,,$$

they become

$$\delta x_/=\frac{dR}{dx}dt,\quad \delta y_/=\frac{dR}{dy}dt,\quad \delta z_/=\frac{dR}{dz}dt\,.$$

Since $(dx_/)$, $(dy_/)$, $(dz_/)$, are supposed to satisfy these equations when $R=0$.

If the preceding values $\delta x_/$, $\delta y_/$, $\delta z_/$, be put in equation (105), it becomes identical with equation (104). Hence the integral (103) satisfies the equations (87), whether the disturbing forces be included or not, the only difference being that, in the first case, *a* must be regarded as a variable quantity, and in the last it is constant.

The same may be shown of all the first integrals of equations (87), when R is zero.

418. It appears, from what has been said, 1st, that as the motion is performed in the unvaried ellipse during the first element of time, *x*, *y*, *z*, *dx*, *dy*, *dz*, are alike in the varied and unvaried ellipse. 2nd, That as the motion is performed in the variable ellipse during the second element of time, if d^2x, d^2y, d^2z, be considered as belonging to the unvaried ellipse, d^2x+ddx, d^2y+ddy, d^2z+ddz will belong to the variable orbit of m. Hence the differential equation of the first order, which determines the motion of the body, answers for both orbits during the first instant of the time, the elements of the orbit being constant; in the second increment of time, the equations of elliptical motion have the form

$$\frac{d^2v}{dt^2}+n^2v=0,$$

the elements of the orbit being constant; but in the troubled orbit they have the form

$$\frac{d^2v}{dt^2}+n^2v+R=0,$$

where the elements of the orbit are variable, and R is the part containing the disturbing forces.

419. As the elements of the orbits only vary during the second increment of the time, their variation is of the first order; that is, the eccentricity e becomes $e+de$, the inclination f becomes

$$f+df, \text{ \&c. \&c.}$$

420. The elegant theory of the variation of the arbitrary constant quantities is due to Euler.[4] Lagrange first applies it to the celestial motions.

421. It is proposed, first, to determine the periodic and secular variations of the elements of orbits of any eccentricities and inclinations; in the second place, to find those of the planets and satellites, all of which have nearly circular orbits, slightly inclined to the plane of the ecliptic; and then to determine the periodic inequalities, dv, ds, dr, in longitude, latitude, and distance.

Variation of the Elements, whatever the Eccentricities and Inclinations may be

422. All the elements of the orbit have been determined from the seven arbitrary constant quantities, c, c', c'', f, f', f'', and a, introduced by the integration of the equations (87) of elliptical motion; but it was shown that the elements of the orbit, as well as the differentials dx, dy, dz, vary during the second element of time by the action of the disturbing forces, and then the differentials of the equations (91) will afford the means of finding the variations of the elements, whatever the eccentricities and inclinations of the orbits may be. Equations (87) give[5]

$$d^2x=dt^2\left(\frac{dR}{dx}\right);\quad d^2y=dt^2\left(\frac{dR}{dy}\right);\quad d^2z=dt^2\left(\frac{dR}{dz}\right);$$

which are the changes in dx, dy, dz, due to the disturbing forces alone, the elliptical part being omitted. If, therefore, the differentials of equations (91) be taken, considering c, c', c'', f, f', f'', a, dx, dy, dz, alone as variable, when the preceding values of d^2x, d^2y, d^2z, are substituted, they become[6]

$$dc = dt\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\};$$

$$dc' = dt\left\{z\left(\frac{dR}{dx}\right) - x\left(\frac{dR}{dz}\right)\right\};$$

$$dc'' = dt\left\{y\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dy}\right)\right\};$$

$$df = dx\left\{z\left(\frac{dR}{dx}\right) - x\left(\frac{dR}{dz}\right)\right\} - dy\left\{y\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dy}\right)\right\} + c'dt\left(\frac{dR}{dx}\right) - c''dt\left(\frac{dR}{dy}\right); \qquad \textbf{(106)}$$

$$df' = dz\left\{y\left(\frac{dR}{dz}\right) - z\left(\frac{dR}{dy}\right)\right\} - dx\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} + c''dt\left(\frac{dR}{dz}\right) - cdt\left(\frac{dR}{dx}\right);$$

$$df'' = dy\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} - dz\left\{z\left(\frac{dR}{dx}\right) - x\left(\frac{dR}{dz}\right)\right\} + cdt\left(\frac{dR}{dy}\right) - c'dt\left(\frac{dR}{dz}\right);$$

$$d\cdot\frac{m}{a} = -2\mathrm{d}R\,.$$

423. If values of c, c', c'', f, f', f'', derived from these equations, be substituted instead of their constant values in equations

$$\tan f = \frac{\sqrt{c'^2 + c''^2}}{c}, \quad \tan q = -\frac{c''}{c'},$$

$$h^2 = ma\left(1 - e^2\right) = c^2 + c'^2 + c''^2,$$

$$\tan v_{/} = \frac{f'}{f''}, \text{ and } me = \sqrt{f^2 + f'^2 + f''^2},$$

given in article 374 and those following, they will determine the elements of the disturbed orbit.

The equations

$$c''x + c'y + cz = 0\,,$$

$$mr - h^2 + f''x + f'y + fz = 0\,;$$

and their differentials

$$c''dx + c'dy + cdz = 0\,,$$

$$mdr + f''dx + f'dy + fdx = 0\,,$$

will also answer in the disturbed orbit, provided the same substitution be made.

424. The mean distance a gives the mean motion of m, or more correctly that in the disturbed orbit, which corresponds with the mean motion in the elliptical orbit; for

$$n = a^{-\frac{3}{2}}\sqrt{m}\ .$$

If z be the mean motion of m, then in the elliptical orbit,

$$dz = ndt;$$

but this equation also answers for the disturbed orbit, since the two orbits coincide during the first instant of time. But

$$ddz = dndt, \quad dn = \frac{3an}{2m}\cdot d\,\frac{m}{a};$$

and as the last of equations (106) is[7]

$$d.\frac{m}{a} = -2\mathrm{d}R, \text{ so } dn = -\frac{3an}{m}\mathrm{d}R\,;$$

hence

$$ddz = -\frac{3andt\,.\,\mathrm{d}R}{m};$$

the integral of which is

$$z = -\frac{3}{m}\iint andt.\mathrm{d}R\ . \qquad \textbf{(107)}$$

425. The seven arbitrary constant quantities are only equivalent to five in consequence of the two equations

$$0 = fc + f'c' + f''c''\,,$$
$$0 = \frac{m}{a} + \frac{f^2 + f'^2 + f''^2 - m^2}{c^2 + c'^2 + c''^2}\,.$$

These also exist in the disturbed orbit, when the arbitrary quantities are replaced by their variable values.

426. Since R is given in article 347, all the elements of the disturbed orbit are determined with the exception of $\in$, the longitude of the planet at the epoch. From the equations

$$dv = \frac{hdt}{r^2}, \quad r^2 = \frac{a^2\left(1-e^2\right)^2}{\left(1+e\cos\left(v-v\ \right)\right)^2}\,,$$

it is evident that,

$$dv\cdot\frac{\left(1-e^2\right)^2}{\left(1+e\cos\left(v-v\ \right)\right)^2}=\frac{h}{a^2}dt\,.$$

But

$$h=\sqrt{ma\left(1-e^2\right)}\,;$$

hence

$$\frac{h}{a^2}=a^{-\frac{3}{2}}\sqrt{m}\sqrt{\left(1-e^2\right)}=n\sqrt{1-e^2}\,;$$

therefore

$$ndt=dv\cdot\frac{\left(1-e^2\right)^{\frac{3}{2}}}{\left(1+e\cos\left(v-v\ \right)\right)^2}\,.$$

If

$$\frac{c^{(v-v\ )\sqrt{-1}}+c^{-(v-v\ )\sqrt{-1}}}{2}$$

be put for

$$\cos\left(v-v\ \right),$$

[then]

$$\frac{\sqrt{1-e^2}}{1+e\cos\left(v-v\ \right)}=\frac{2\sqrt{1-e^2}}{2+e\left\{c^{(v-v\ )\sqrt{-1}}+c^{-(v-v\ )\sqrt{-1}}\right\}}\,.$$

Again, if

$$l=\frac{e}{1+\sqrt{1-e^2}};\ \text{then}\ e=\frac{2l}{1+l^2}\,,$$

which, substituted in the second member of the last equation, gives

$$\frac{1}{1+e\cos\left(v-v\ \right)}=\frac{1}{\sqrt{1-e^2}}\left\{\frac{1-l^2}{1+l^2+l\left\{c^{(v-v\ )\sqrt{-1}}+c^{-(v-v\ )\sqrt{-1}}\right\}}\right\}.$$

The numerator of the last term is

$$1-l^2=\left(1+l\,c^{-(v-v\ )\sqrt{-1}}\right)-l\,c^{-(v-v\ )\sqrt{-1}}\left(1+l\,c^{(v-v\ )\sqrt{-1}}\right)$$

And the denominator is equal to

$$\left(1+l\,c^{(v-v\ )\sqrt{-1}}\right)\left(1+l\,c^{-(v-v\ )\sqrt{-1}}\right)$$

hence

$$\frac{1}{1+e\cos(v-v\ )}=\frac{1}{\sqrt{1-e^2}}\left\{\frac{1}{1+lc^{(v-v\ )\sqrt{-1}}}-\frac{lc^{-(v-v\ )\sqrt{-1}}}{1+lc^{-(v-v\ )\sqrt{-1}}}\right\}.$$

By division,

$$\frac{1}{1+lc^{(v-v\ )\sqrt{-1}}}=1-lc^{(v-v\ )\sqrt{-1}}+l^2c^{2(v-v\ )\sqrt{-1}}-\&\text{c.}$$

[and]

$$\frac{lc^{-(v-v\ )\sqrt{-1}}}{1+lc^{-(v-v\ )\sqrt{-1}}}=lc^{-(v-v\ )\sqrt{-1}}-l^2c^{-2(v-v\ )\sqrt{-1}}+\&\text{c.}$$

And the difference of these is[8]

$$\frac{1}{1+e\cos(v-v\ )}=\frac{1}{\sqrt{1+e^2}}\left\{1-l\left(c^{(v-v\ )\sqrt{-1}}+c^{-(v-v\ )\sqrt{-1}}\right)+l^2\left(c^{2(v-v\ )\sqrt{-1}}+c^{-2(v-v\ )\sqrt{-1}}\right)-\&\text{c.}\right\};$$

but

$$c^{i(v-v\ )\sqrt{-1}}+c^{-i(v-v\ )\sqrt{-1}}=2\cos i(v-v\ );$$

hence

$$\frac{1}{1+e\cos(v-v\ )}=\frac{1}{\sqrt{1+e^2}}\left\{1-2l\cos(v-v\ )+2l^2\cdot\cos 2(v-v\ )-\&\text{c.}\right\}$$

or

$$\frac{1}{1+e\cos(v-v\ )}=\frac{1}{\sqrt{1+e^2}}\mp 2\cos i(v-v\ )\frac{l^i}{\sqrt{1-e^2}},$$

which is the general form of the series, *i* being any whole positive number.

Now,

$$\frac{1}{de}\cdot d\frac{e}{1+e\cos(v-v\ )}=\frac{1}{\left(1+e\cos(v-v\ )\right)^2}=\frac{1}{de}\cdot\left\{d\frac{e}{\sqrt{1-e^2}}\pm 2\cos(v-v\ )\cdot d\frac{el^i}{\sqrt{1-e^2}}\right\};$$

but

$$d\cdot\frac{e}{\sqrt{1-e^2}}=\frac{de}{\left(1-e^2\right)^{\frac{3}{2}}},\ \text{and}\ d\frac{el^i}{\sqrt{1-e^2}}=\pm\frac{e^i\left\{1+i\sqrt{1-e^2}\right\}de}{\left(1-e^2\right)^{\frac{3}{2}}\left(1+\sqrt{1-e^2}\right)^i}$$

the sign + is used here when *i* is even, and − when it is odd. Hence if to abridge

$$\mathrm{E}^{(i)}=\pm\frac{2e^i\cdot\left\{1+i\sqrt{1-e^2}\right\}}{\left(1+\sqrt{1-e^2}\right)^i},$$

the value of *ndt* becomes,

$$ndt = dv\left\{1 + E^{(1)}\cos(v - \varpi) + E^{(2)}\cos 2(v - \varpi) + \&c.\right\}; \qquad \textbf{(108)}$$

The integral of which is[9]

$$\int ndt + \epsilon = v + E^{(1)}\sin(v - \varpi) + \tfrac{1}{2}E^{(2)}\sin 2(v - \varpi) + \&c.,$$

ϵ being arbitrary.

This equation is relative to the invariable ellipse; but in order that it may also suit the real orbit, every quantity in it must vary including *e*, ϖ, and ϵ; and this differential must coincide with (108) since they are of the first order, and the two orbits coincide during the first element of time. Their difference is

$$d\epsilon = de\left\{\left(\frac{dE^{(1)}}{de}\right)\sin(v - \varpi) + \tfrac{1}{2}\left(\frac{dE^{(2)}}{de}\right)\sin 2(v - \varpi) + \&c.\right\}$$
$$- d\varpi\left\{E^{(1)}\cos(v - \varpi) + E^{(2)}\cos 2(v - \varpi) + \&c.\right\}$$

$v - \varpi$ is the true anomaly of *m* estimated on the orbit, and ϖ is the longitude of the perihelion on the orbit. Now equation (101) is[10]

$$v - \xi = v_/ - \theta + \tan^2\tfrac{1}{2}\phi \cdot \sin 2(v_/ - \theta) + \tfrac{1}{2}\tan^4\tfrac{1}{2}\phi \cdot \sin 4(v_/ - \xi) + \&c.$$

v being the longitude on the orbit, and $v_/$ its projection on the fixed plane. If ϖ be put for *v* and $\varpi_/$ for $v_/$; then

$$\varpi - \xi = \varpi_/ - \theta + \tan^2\tfrac{1}{2}\phi\sin 2(\varpi_/ - \theta) + \&c.$$

Again, if we make *v* and $v_/$ zero in equation (101), it becomes

$$\xi = \theta + \tan^2\tfrac{1}{2}\phi\sin 2\theta + \tfrac{1}{2}\tan^4\tfrac{1}{2}\phi\sin 4\theta + \&c.$$

hence

$$\varpi = \varpi_/ + \tan^2\tfrac{1}{2}\phi\left\{\sin 2\theta + \sin 2(\varpi_/ - \theta)\right\} + \&c.$$

therefore

$$d\varpi = d\varpi_/\left\{1 + 2\tan^2\tfrac{1}{2}\phi\cos 2(\varpi_/ - \theta) + \&c.\right\}$$
$$+ 2d\theta\tan^2\tfrac{1}{2}\phi\left\{\cos 2\theta - \cos 2(\varpi_/ - \theta) + \&c.\right\}$$
$$+ \frac{d\phi\tan\tfrac{1}{2}\phi}{\cos^2\tfrac{1}{2}\phi}\left\{\sin 2\theta + \sin 2(\varpi_/ - \theta) + \&c.\right\}$$

Thus $dv_{/}$, dq, df, being determined, we shall have dv from this equation, and from thence $d\in$.

427. It appears from the preceding investigations, that the expressions in series given by the equations in article 392, and those following, of the radius vector, of its projection on the fixed plane, of the longitude, and its projection on the fixed plane, and of the latitude in the invariable orbit will answer for the disturbed orbit, provided *nt* be changed into $\int ndt$, and all the elements of the variable orbit be determined by the preceding equations; for the finite equations between *r*, *v*, *s*, *x*, *y*, *z*, and $\int ndt$, are the same in both cases, and all the equations in the articles alluded to are determined independently of the constancy or variation of the elements, consequently these expressions will still answer when the elements are variable.

These investigations relate to orbits of any inclination and eccentricity; but the orbits of the planetary system are nearly circular, and very little inclined either to one another, or to the plane of the ecliptic.

Variations of the Elliptical Elements of the Orbits of the Planets

428. The equation

$$n = a^{-\frac{3}{2}}\sqrt{m}$$

shows that the mean motions and greater axes of the orbits of the planets are so connected, that one cannot vary independently of the other; and as

$$\frac{m}{a} = -2\int \mathrm{d}R\,,$$

it is clear that the differential of R is taken only with regard to *nt* the mean motion of *m*. If the mass of the sun be assumed as the unit, and the mass of the planet omitted in comparison of it, $m = 1$, and

$$da = 2a^2\mathrm{d}R\,;$$

2*a* being the major axis.

429. The inequalities in the eccentricity and longitude of the perihelion are obtained from

$$\tan v_{/} = \frac{f'}{f''},\ \ me = \sqrt{f^2 + f'^2 + f''^2}$$

$v_{/}$ being the longitude of the perihelion of *m* when projected on the fixed plane of the ecliptic. If the orbit of the planet *m* at a given epoch be assumed to be the fixed plane containing the axes *x* and *y*, any inclination the orbit may have at a subsequent period being entirely owing to the action of the disturbing forces must be so small, that the true longitude of the perihelion will only differ from its projection on that new fixed plane, by quantities of the order of the squares of the disturbing masses respectively multiplied by the squares of the inclinations of the orbits, therefore without sensible error it may be assumed that $v_{/} = v$; v being the longitude of the perihelion estimated on the orbit; thus

$$\tan v = \frac{f'}{f''},$$

whence

$$\sin v = \frac{f'}{\sqrt{f'^2 + f''^2}};$$

and

$$\cos v = \frac{f''}{\sqrt{f'^2 + f''^2}}.$$

But by article 370 $f = -\frac{f'c' + f''c''}{c}$. Now c, c', c'' are the areas described by the radius vector of m on its orbit, when projected on the co-ordinate planes; but as the orbit nearly coincides with the fixed plane of the orbit at the epoch containing the axes x and y, the other two co-ordinate planes are nearly at right angles to it; hence c', and c'' are extremely small, and as f is of the same order in consequence of the preceding equation it may be omitted, so that

$$e = \sqrt{f''^2 + f'^2}$$

whence

$$f'' = e\cos v; \quad f' = e\sin v ,$$

and

$$ede = f''df'' + f'df'; \quad e^2dv = f''df' - f'df'',$$

making $m = 1$.

430. Since f is very small df is still smaller, therefore the fourth of the equations (91) may be omitted as well as $c'dt = zdx - xdz$, and $c''dt = ydz - zdy$, on account of the smallness of c' and c''. Also z, the height of the planet above the fixed plane of its orbit, is so small that its square may be neglected; therefore quantities having the factors zdz, or $dz\left(\frac{dR}{dz}\right)$ may be omitted, which reduces the values of the fifth and sixth of equations (106) to

$$df'' = dy\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} + cdt\left(\frac{dR}{dy}\right),$$

[and]

$$df' = -dx\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} - cdt\left(\frac{dR}{dx}\right).$$

431. If $r_{/} = Sp$, fig. 77, be the radius vector of *m* projected on the fixed plane of the orbit of *m* containing the axes *x* and *y*; and if the angle N*Sp* be represented by $v_{/}$, and *pm* the tangent of the latitude of *m* above the fixed plane of its orbit by *s*, then

$$x = r_{/}\cos v_{/};\ y = r_{/}\sin v_{/};\ z = r_{/}s\,.$$

Since *x* is a function of $r_{/}$ and $v_{/}$,

$$\frac{dR}{dx} = \frac{dR}{dr_{/}}\cdot\frac{dr_{/}}{dx}\,,$$
$$\frac{dR}{dx} = \frac{dR}{dv_{/}}\cdot\frac{dv_{/}}{dx}\,.$$

But

$$\frac{dr_{/}}{dx} = \frac{1}{\cos v_{/}};\quad \frac{dv_{/}}{dx} = -\frac{1}{r_{/}\sin v_{/}};$$

hence

$$\frac{dR}{dx} = \frac{dR}{dr_{/}}\cdot\frac{1}{\cos v_{/}};\quad \frac{dR}{dx} = -\frac{dR}{dv_{/}}\cdot\frac{1}{r_{/}\sin v_{/}}\,.$$

If the first equation be multiplied by $\cos^2 v_{/}$, and the second by $\sin^2 v_{/}$, their sum will be,[11]

$$\frac{dR}{dx} = \left(\frac{dR}{dr_{/}}\right)\cos v_{/} - \left(\frac{dR}{dv_{/}}\right)\frac{\sin v_{/}}{r_{/}}\,.$$

In like manner it may be found that

$$\frac{dR}{dy} = \left(\frac{dR}{dr_{/}}\right)\sin v_{/} + \left(\frac{dR}{dv_{/}}\right)\frac{\cos v_{/}}{r_{/}};$$

whence

$$x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right) = \frac{dR}{dv_{/}};$$

consequently,[12]

$$df'' = +dy\left(\frac{dR}{dv_{/}}\right) + cdt\left\{\left(\frac{dR}{dr_{/}}\right)\sin v_{/} + \left(\frac{dR}{dv_{/}}\right)\frac{\cos v_{/}}{r_{/}}\right\},$$
$$df' = -dx\left(\frac{dR}{dv_{/}}\right) - cdt\left\{\left(\frac{dR}{dr_{/}}\right)\cos v_{/} - \left(\frac{dR}{dv_{/}}\right)\frac{\sin v_{/}}{r_{/}}\right\};$$

but

$$dx = d\left(r_{/}\cos v_{/}\right);\ \ dy = d\left(r_{/}\sin v_{/}\right),$$

and

$$cdt = xdy - ydx = r_{/}^{2}dv_{/};$$

so that[13]

$$df'' = +\left\{dr_{/}\sin v_{/} + 2r_{/}dv_{/}\cos v_{/}\right\}\left(\frac{dR}{dv_{/}}\right) + r_{/}^{2}dv_{/}\sin v_{/}\left(\frac{dR}{dr_{/}}\right),$$

$$df' = -\left\{dr_{/}\cos v_{/} - 2r_{/}dv_{/}\sin v_{/}\right\}\left(\frac{dR}{dv_{/}}\right) - r_{/}^{2}dv_{/}\cos v_{/}\left(\frac{dR}{dr_{/}}\right).$$

432. The values of $r_{/}$, $dr_{/}$, $dv_{/}\left(\frac{dR}{dr_{/}}\right)$, $\left(\frac{dR}{dr_{/}}\right)$, are the same from whatever point the longitudes may be estimated; but by diminishing the angle $v_{/}$ by a right angle, $\sin v_{/}$ becomes $-\cos v_{/}$; and $\cos v_{/}$ becomes $\sin v_{/}$, so that the expression of df'' is changed into that of df', whence it follows, that if the value of df'' be developed into a series of sines and cosines of angles increasing proportionally with the time, and if each of the angles $\in$, $\in'$, v, v', q, q', be diminished by 90°, the value of df' will be obtained.

433. By articles 398 and 401, the projection of the longitude on the fixed plane of the ecliptic, and the curtate distance are,[14]

$$v_{/} - q = v - x - \tan^{2}\tfrac{1}{2}f\sin 2\left(v - x\right) + \&c.$$

$$r_{/} = r\left\{1 - \tfrac{1}{2}s^{2} + \&c.\right\}.$$

But when the orbit of m at the epoch is assumed to be the fixed plane, any inclination it may have at a subsequent period, arises entirely from the action of the disturbing forces, and is so very small that the squares of the tangent of that inclination may be neglected, whence,

$$v_{/} - q = v - x,\ \ r_{/} = r,\ \ v_{/} = v, \text{ and } q = x.$$

In the invariable orbit,[15]

$$r = \frac{a\left(1-e^{2}\right)}{1+e\cos\left(v-v\right)},\ \ dr = \frac{r^{2}dv\cdot e\sin\left(v-v\right)}{a\left(1-e^{2}\right)},\ \ r^{2}dv = a^{2}.n.dt\sqrt{1-e^{2}}.$$

But these equations answer also for the variable orbit, since the two ellipses coincide during the first element of time, and when substitution is made for r, dr, and $r^{2}dv$ in the last values of df'' and df', they become

$$df'' = \frac{a\,.\,ndt}{\sqrt{1-e^2}}\left\{2\cos v + \tfrac{3}{2}e\cos v + \tfrac{1}{2}\cos(2v - v)\right\}\left(\frac{dR}{dv}\right) + a^2\cdot ndt\sqrt{1-e^2}\,\sin v\left(\frac{dR}{dr}\right),$$

$$df' = \frac{a\,.\,ndt}{\sqrt{1-e^2}}\left\{2\sin v + \tfrac{3}{2}e\sin v + \tfrac{1}{2}\sin(2v - v)\right\}\left(\frac{dR}{dv}\right) - a^2\cdot ndt\sqrt{1-e^2}\,\cos v\left(\frac{dR}{dr}\right).$$

But

$$f'' = e\cos v\,,\quad f' = e\sin v$$

and by means of these equations the expressions

$$ede = f''df'' + f'df'$$

and

$$e^2dv = f''df' - f'df''$$

in consequence of

$$\cos(2v - 2v) = 2\cos^2(v - v) - 1,$$

become[16]

$$de = \frac{a\cdot ndt}{\sqrt{1-e^2}}\left\{2\cos(v - v) + e + e\cos^2(v - v)\right\}\cdot\left(\frac{dR}{dv}\right) + a^2\cdot ndt\sqrt{1-e^2}\,\sin(v - v)\left(\frac{dR}{dr}\right), \qquad \textbf{(109)}$$

$$edv = -\frac{a\cdot ndt}{\sqrt{1-e^2}}\sin(v - v)\left\{2 + e\cos(v - v)\right\}\cdot\left(\frac{dR}{dv}\right) - a^2ndt\sqrt{1-e^2}\,\cos(v - v)\left(\frac{dR}{dr}\right). \qquad \textbf{(110)}$$

The variation of the eccentricity however may be obtained under a more simple form from the equation $c = \sqrt{ma\left(1-e^2\right)}$ article 422, c' and c'' being zero, for

$$de = \frac{da\sqrt{a\left(1-e^2\right)}}{2a} - \frac{ede\sqrt{a}}{\sqrt{1-e^2}};$$

but

$$\frac{de}{dt} = x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dv}\right);$$

hence by comparing the two values of *de*, and observing that $\frac{da}{2a^2} = \mathrm{d}R$,

$$ede = -a \cdot ndt\sqrt{1-e^2}\left(\frac{dR}{dv}\right) + a\left(1-e^2\right)\mathrm{d}R. \qquad \textbf{(111)}$$

434. The variation in the longitude of the epoch may be found by the preceding equations (109) and (110). For it was shown in article 392, that if the mean anomaly be estimated from any other point than the perihelion, $nt + \in - v$ may be put for nt, or rather $\int ndt + \in - v$; hence the equations in article 385 are

$$\int ndt + \in - v = u - e\sin u\,,$$

$$r = a\left(1-e\cos u\right),$$

$$\tan\tfrac{1}{2}\left(v - v\right) = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}u\,,$$

and

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(v-v\right)}.$$

In the invariable orbit,

$$ndt = du\left(1-e\cos u\right),$$

in which u varies with the time. But if we suppose the time constant, and u to vary only in consequence of the variation of e and v, then in the troubled orbit,

$$d\in - dv = du\left(1-e\cos u\right) - de\sin u\,.$$

From the third of the preceding equations,

$$-\frac{dv}{\cos^2\frac{1}{2}\left(v-v\right)} = \frac{du}{\cos^2\frac{1}{2}u}\cdot\sqrt{\frac{1+e}{1-e}} + \frac{2de\tan\frac{1}{2}u}{(1-e)\sqrt{1-e^2}}$$

and substituting for $\cos^2\frac{1}{2}\left(v-v\right)$, its value from the same equation, the result is

$$du = -\frac{dv\,\left(1-e\cos u\right)}{\sqrt{1-e^2}} - \frac{de\sin u}{1-e^2};$$

hence

$$d\in - dv = -\frac{dv\,\left(1-e\cos u\right)^2}{\sqrt{1-e^2}} - \frac{de\sin u\left(2-e^2-e\cos u\right)}{1-e^2};$$

or

$$d\in - dv + dv\sqrt{1-e^2} = \frac{edv}{\sqrt{1-e^2}}\left\{2\cos u - e - e\cos^2 u\right\} - \frac{de}{1-e^2}\sin u\left(2-e^2-e\cos u\right).$$

Now[17]

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(v-v\right)} = a\left(1-e\cos u\right),$$

whence

$$\cos u = \frac{e+\cos\left(v-v\right)}{1+e\cos\left(v-v\right)}, \quad \sin u = \frac{\sqrt{1-e^2}\sin\left(v-v\right)}{1+e\cos\left(v-v\right)}.$$

And substituting these,

$$d\in - dv\left(1-\sqrt{1-e^2}\right) = \sqrt{1-e^2}\,\frac{\left\{2\cos\left(v-v\right)+e+e\cos^2\left(v-v\right)\right\}}{\left(1+e\cos\left(v-v\right)\right)^2}\cdot edv$$
$$-\sqrt{1-e^2}\,\frac{\left\{2+e\cos\left(v-v\right)\right\}}{\left\{1+e\cos\left(v-v\right)\right\}^2}de\sin\left(v-v\right).$$

If the values of edv and de, given by equations (109) and (110), be substituted, the result will be

$$d\in = dv\left(1-\sqrt{1-e^2}\right) - 2a\cdot ndt\cdot r\left(\frac{dR}{dr}\right);$$

but

$$r\left(\frac{dR}{dr}\right) = a\left(\frac{dR}{da}\right);$$

hence

$$d\in = dv\left(1-\sqrt{1-e^2}\right) - 2a^2\left(\frac{dR}{da}\right)\cdot ndt,$$

which is the variation in the epoch.

435. The variations in the inclination of the orbits, and in the longitude of their nodes, are obtained from[18]

$$\tan f = \frac{\sqrt{c'^2+c''^2}}{c}, \quad \tan q = \frac{-c''}{c'},$$
$$\text{for } \tan f\cos q = -\frac{c'}{c}; \quad \tan f\sin q = \frac{c''}{c};$$

whence[19]

$$d\,.\tan f = \frac{1}{c}\left\{dc''\sin q - dc'\cos q - dc\tan f\right\},$$

$$dq \cdot \tan f = \frac{1}{c}\left\{dc'' \cos q + dc' \sin q\right\}.$$

If substitution be made for[20] $\frac{dc}{dt}, \frac{dc'}{dt}, \frac{dc''}{dt},$ of their values in article 422, and making

$$x = r\cos v, \quad y = r\sin v,$$
$$s = \tan f \sin(v - q),$$

there will result[21]

$$\begin{aligned} d \cdot \tan f &= -\frac{dt \tan f \cos(v-q)}{c}\left\{r\left(\frac{dR}{dr}\right)\sin(v-q) + \left(\frac{dR}{dv}\right)\cos(v-q)\right\} \\ &\quad + \frac{(1+s^2)\,dt}{c}\cos(v-q)\left(\frac{dR}{ds}\right) \\ dq \cdot \tan f &= -\frac{dt \tan f \sin(v-q)}{c}\left\{r\left(\frac{dR}{dr}\right)\sin(v-q) + \left(\frac{dR}{dv}\right)\cos(v-q)\right\} \\ &\quad + \frac{(1+s^2)\,dt}{c}\sin(v-q)\left(\frac{dR}{ds}\right). \end{aligned} \qquad \textbf{(112)}$$

These two equations determine the inclination of the orbit, and motion of the nodes. They give[22]

$$\sin(v-q) \cdot d\tan f - dq \cdot \cos(v-q)\tan f = 0,$$

which may also be obtained from

$$s = \tan f \sin(v-q).$$

436. If the orbit of m has so small an inclination on the fixed plane, that the squares of s and $\tan f$ may be omitted, then

$$d \cdot \tan f = \frac{dt}{c}\cos(v-q)\left(\frac{dR}{ds}\right),$$
$$dq \cdot \tan f = \frac{dt}{c}\sin(v-q)\left(\frac{dR}{ds}\right);$$

if to abridge

$$p = \tan f \sin q, \quad q = \tan f \cos q,$$

and as

$$c = \sqrt{a(1-e^2)}; \quad a = \frac{1}{a^2 n^2};$$

[then]

$$\frac{1}{c}=\frac{an}{\sqrt{1-e^2}};$$

these become[23]

$$dp=\frac{andt}{\sqrt{1-e^2}}\sin v\left(\frac{dR}{ds}\right),$$
$$dq=\frac{andt}{\sqrt{1-e^2}}\cos v\left(\frac{dR}{ds}\right).$$

But

$$z=+qy-px;$$

and as the orbit is supposed to have a very small inclination on the fixed plane, $r\cos v$, $r\sin v$, and rs, may be put for x, y, and z, the last equation becomes

$$s=q\sin v-p\cos v,$$

whence

$$\frac{dR}{ds}=\frac{1}{\sin v}\left(\frac{dR}{dq}\right);\quad \frac{dR}{ds}=-\frac{1}{\cos v}\left(\frac{dR}{dp}\right);$$

consequently

$$dq=-\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dp}\right)$$
$$dp=\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dq}\right).$$

437. But when the inclination of the orbit is very small,

$$\frac{z}{a}=q\sin(nt+\epsilon)-p\cos(nt+\epsilon)$$

whence[24]

$$dp=-\frac{a^2ndt}{\sqrt{1-e^2}}\sin(nt+\epsilon)\left(\frac{dR}{dz}\right),$$
$$dq=\frac{a^2ndt}{\sqrt{1-e^2}}\cos(nt+\epsilon)\left(\frac{dR}{dz}\right);$$

for[25]

$$\frac{dR}{dp}=-\left(\frac{dR}{dz}\right)\cos(nt+\epsilon),$$
$$\frac{dR}{dq}=\left(\frac{dR}{dz}\right)\sin(nt+\epsilon);$$

and

$$x=a\cos(nt+\epsilon),\quad y=a\sin(nt+\epsilon).$$

438. Since the elliptical and troubled orbits coincide during the first element of the time, the equations of motion are identical for that period, therefore the variation of the elements must be zero; consequently,

$$0=\left(\frac{dR}{da}\right)da+\left(\frac{dR}{de}\right)de+\left(\frac{dR}{dv}\right)dv+\left(\frac{dR}{d\in}\right)d\in+\left(\frac{dR}{dp}\right)dp+\left(\frac{dR}{dq}\right)dq \qquad \textbf{(113)}$$

Because nt is always accompanied by $-v$, therefore

$$\frac{dR}{dv}=\frac{dR}{ndt}+\frac{dR}{dv},$$

so that the differential *de* becomes

$$de=-andt\frac{\sqrt{1-e^2}}{e}\cdot\left(\frac{dR}{dv}\right)-a\frac{\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\mathrm{d}R.$$

If this value of de, and the preceding values of da, $d\in$, dp, dq, be substituted in equation (113), observing that $\frac{dR}{ndt}$ may be put for $\frac{dR}{d\in}$ and $\frac{dR}{dv}$, it will be reduced to

$$dv=\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{de}\right);$$

whence

$$d\in=\frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\cdot\left(\frac{dR}{de}\right)-2a^2\left(\frac{dR}{da}\right)ndt.$$

By article 424,

$$dz=-3\int andt\,.\,\mathrm{d}R;$$

the integral of which is the periodic inequality in the mean motion.

439. The differential equations of the periodic variations of the elements of the orbit of *m* are therefore

$$da=2a^2\mathrm{d}R;$$
$$dz=-3\int andt\,.\,\mathrm{d}R;$$
$$d\in=\frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left(\frac{dR}{de}\right)-2a^2\left(\frac{dR}{da}\right)ndt;$$

$$de = -\frac{a\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\mathrm{d}R - \frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{d\varpi}\right); \qquad \textbf{(114)}$$

$$d\varpi = \frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{de}\right);$$

$$dp = \frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dq}\right);$$

$$dq = -\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dp}\right).$$

Because $\in$ always accompanies nt,

$$\frac{dR}{d\in} = \frac{\mathrm{d}R}{ndt}; \text{ whence } ndt\left(\frac{dR}{d\in}\right) = \mathrm{d}R;$$

so that *da* may also be expressed by

$$da = 2a^2 ndt\left(\frac{dR}{d\in}\right).$$

440. By article 347, R is a given function of x, y, z, x', y', z', &c., the co-ordinates m, m', m'', &c. and is of the first order with regard to the masses; and if the squares and products of the masses be omitted, the elliptical values of x, y, z, x', y', z', &c. may be substituted, and then R will be a function of the time, and of the elements of the orbits, and may therefore be developed in a series of sines and cosines containing the time. But the first part of this series is independent of the time, being a function of the elements of the orbits alone, as will be shown immediately, and may be represented by F.

441. As F does not contain the arc nt, its differential with regard to that quantity, is zero, consequently when F is put for R in the preceding equations they become[26]

$$da = 0; \quad dz = 0;$$

$$d\in = \frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left(\frac{dF}{de}\right) - 2a^2 ndt\left(\frac{dF}{da}\right);$$

$$de = -\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dF}{d\varpi}\right);$$

$$d\varpi = \frac{andt\sqrt{1-e^2}}{e}\left(\frac{dF}{de}\right); \qquad \textbf{(115)}$$

$$dp = \frac{andt}{\sqrt{1-e^2}}\left(\frac{dF}{dq}\right);$$

$$dq = -\frac{andt}{\sqrt{1-e^2}}\left(\frac{dF}{dp}\right).$$

The integrals of these equations are the secular variations the elements of the orbit of m.

442. In the determination of the periodic variations of the elements, all terms of the series *R*, that do not contain the time, must be omitted; and in the secular variations, all terms of that series that do contain the time must be rejected. Thus the periodic variations in the elements of the planetary orbits depend on the configuration, or relative position of the bodies, and their secular variations do not.

443. These periodic and secular variations, in the elements of elliptical motion, are sufficient for the determination of all the inequalities to which the bodies of the solar system are liable in their revolutions round the sun. On the same principle, the periodic and secular variations in the rotation of the earth and planets may be found from the variation of the six arbitrary constant quantities introduced by the integration of the equations of rotatory motion. The expressions of these variations are identical in the motions of translation and rotation; and as the perturbations in these two motions arise from the same cause, they are expressed by the same formulae. The analysis by which Lagrange[27] has united the two great problems of the solar system is the most refined and elegant in the science of astronomy.

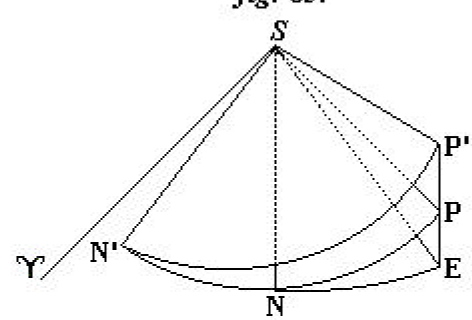
444. Observation shows the inclinations of the orbits of the planets on the plane of the ecliptic to be very small; hence if EN, Fig. 83, be the fixed plane of the ecliptic at a given epoch, PN the orbit of m, $\mathrm{P'N'}$ the orbit of m'[28]

$$\mathrm{ENP} = f, \quad \mathrm{EN'P'} = f',$$

the inclination of these orbits on the plane of the ecliptic; and

$$♈ S\mathrm{N} = q, \quad ♈ S\mathrm{N'} = q',$$

fig. 83.



the longitudes of their ascending nodes on the same plane, then if the planet m were moving on the orbit PN, the tangent of its latitude would be

$$z = \mathrm{EP} = \tan f \sin\left(nt + \epsilon - q\right).$$

And if it were moving on the orbit $\mathrm{P'N'}$, the tangent of its latitude would be

$$z' = \mathrm{EP'} = \tan f' \sin\left(nt + \epsilon - q'\right).$$

Hence if g be the tangent of the inclination of the orbit $\mathrm{P'N'}$ on the orbit PN, and Π the longitude of the line of common intersection of these two planes, or of the ascending node of the orbit of m' on that of m, then

$$\tan f' \sin\left(nt+\in -q'\right) - \tan f \sin\left(nt+\in -q\right) = g \sin\left(nt+\in -\Pi\right) = z' - z = \mathrm{PP'}\text{nearly}.$$

If then as before[29]

$$p = \tan f \sin q, \quad q = \tan f \cos q, \qquad \textbf{(116)}$$
$$p' = \tan f' \sin q', \quad q' = \tan f' \cos q';$$

there will be found[30] [31]

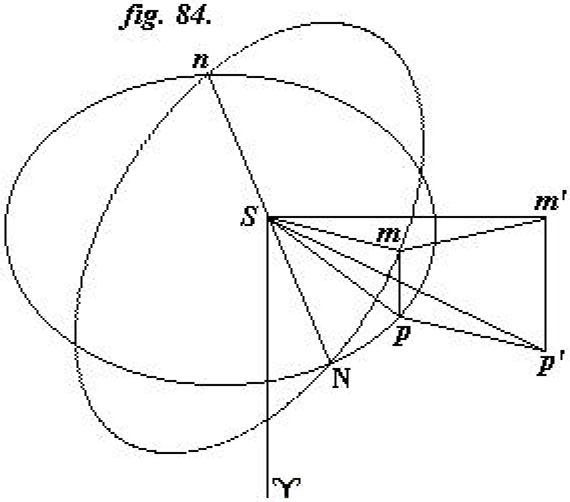
$$l \sin \Pi = p' - p, \quad g \cos \Pi = q' - q, \qquad \textbf{(117)}$$
$$g^2 = \left(p' - p\right)^2 + \left(q' - q\right)^2.$$

Now if EN be the primitive orbit of m at the epoch, and PN its orbit at any other period, $z = 0$, $f = 0$, and $g = \tan f'$; and it is evident that g, the tangent of the mutual inclination of these two planes, will be of the order of the disturbing forces; and therefore very small, since any inclination the orbit may acquire subsequently to the epoch is owing to the disturbing forces.

445. It is now requisite to develop R into a series of the sines and cosines of the mean angular distances of the bodies.

If the disturbing action of only one body be estimated at a time[32]

fig. 84.



$$R = \frac{m'}{\sqrt{\left(x'-x\right)^2 + \left(y'-y\right)^2 + \left(z'-z\right)^2}} - \frac{m'\left(x'x + y'y + z'z\right)}{\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}},$$

in which

$$r = Sm = \sqrt{x^2 + y^2 + z^2}; \quad r' = Sm' = \sqrt{x'^2 + y'^2 + z'^2},$$

$$mm' = \sqrt{\left(x'-x\right)^2 + \left(y'-y\right)^2 + \left(z'-z\right)^2}.$$

The orbits of the planets are nearly circular, and their greatest inclination on the plane of the ecliptic does not exceed 7°, R developed according to the powers and products of these quantities must necessarily be very convergent; but as R is independent of the position of the co-ordinate planes, the plane of projection N*pn*, fig. 84, may be so chosen as to make the inclination still less, consequently z and z' will be very small.

Let $v_{/} = \Upsilon Sp$, $v'_{/} = \Upsilon S\,p'$, be the projected longitudes of m and m' on the fixed plane, and let

$$r_{/} = Sp \quad r'_{/} = Sp'$$

be their curtate distances; then[33]

$$x = r_{/} \cdot \cos v_{/}; \quad y = r_{/} \cdot \sin v_{/};$$
$$x' = r'_{/} . \cos v'_{/}; \quad y' = r'_{/} . \sin v'_{/};$$

hence

$$R = \frac{m'}{\sqrt{r'^2_{/} - 2r_{/}r'_{/} . \cos\left(v'_{/} - v_{/}\right) + r_{/}^2 + \left(z' - z\right)^2}} - \frac{m'\left(r_{/}r'_{/} . \cos\left(v'_{/} - v_{/}\right) + z'z\right)}{\left(r'^2_{/} + z'^2\right)^{\frac{3}{2}}}$$

or z and z' being extremely small,[34]

$$R = \frac{m'}{\sqrt{r'^2_{/} - 2r_{/}r'_{/} . \cos\left(v'_{/} - v_{/}\right) + r_{/}^2}} - \frac{r_{/} \cdot \cos\left(v'_{/} - v_{/}\right) \cdot m'}{r'^2_{/}} - \frac{m' . zz'}{r'^3_{/}}$$
$$+ \frac{3m'r_{/} \cdot z'^2 \cos\left(v'_{/} - v_{/}\right)}{2r'^4_{/}} - \frac{m'\left(z' - z\right)^2}{2\left\{r'^2_{/} - 2r'r'_{/}\cos\left(v'_{/} - v_{/}\right) + r_{/}^2\right\}^{\frac{3}{2}}} + \&c$$

Because the eccentricities and inclinations of the orbits of the planets and satellites are very small, it appears from the values of the radius vector and true longitude in the elliptical orbit developed in article 398, and those following, that

$$r_{/} = a\left(1 + u\right); \quad r'_{/} = a'\left(1 + u'\right);$$
$$v_{/} = nt + \epsilon + v; \quad v'_{/} = n't + \epsilon' + v';$$

u, u', v, v', being very small quantities depending on the eccentricities and inclinations, and a, a' the mean distances of m and m', or half the greater axes of their orbits.

If these quantities be substituted in R, and if to abridge

$$n't - nt + \epsilon' - \epsilon = b\ ,$$

observing also that,

$$\cos\left(b + v' - v\right) = \cos b \cdot \cos\left(v' - v\right) - \sin b \cdot \sin\left(v' - v\right) = \cos b - \left(v' - v\right)\sin b\ ,$$

because $v' - v$ is so small that it may be taken for its sine and unity for its cosine, thus

$$R=-m'\cdot\frac{a}{a'^2}\cdot\frac{1+u}{(1+u')^2}\cdot\cos b+m'\cdot\frac{a}{a'^2}\cdot\frac{1+u}{(1+u')^2}\cdot\sin b$$
$$+\frac{m'}{\left\{a^2(1+u)^2-2aa'(1+u)(1+u').\cos b+a'^2(1+u')^2\right\}^{\frac{1}{2}}}$$
$$-\frac{m'\cdot zz'}{a'^3}+\frac{3m'\cdot az'^2}{2a'^4}\cdot\cos b$$
$$-\frac{m'(z-z')^2-3m'\cdot az'^2(v'-v)\cdot\sin b}{2\left\{a^2(1+u)^2-2aa'(1+u)(1+u').\cos b+a'^2(1+u')^2\right\}^{\frac{3}{2}}}+\text{\&c.}$$

446. The expansion of this function into a series ascending, according to the powers and products of the very small quantities u, u', v, v', z, and z' is easily accomplished by the theorem for the development of a function of any number of variables, for if R' be the value of R when these small quantities are zero, that is, supposing the orbits to be circular and all in one plane, then

$$R=R'+au\cdot\frac{dR'}{da}+a'u'\cdot\frac{dR'}{da'}+(v'-v)\cdot\frac{dR'}{ndt}+\frac{a^2u^2}{2}\cdot\frac{d^2R'}{da^2}+\frac{a'^2u'^2}{2}\cdot\frac{d^2R'}{da'^2}+\text{\&c.}$$

because a is the only quantity that varies with u, a' with u', and t with $(v'-v)$. But

$$R'=m'\left\{(a^2-2aa'\cos b+a'^2)^{-\frac{1}{2}}-\frac{a}{a'^2}\cos b\right\};$$

and if [35] $(a'^2-2aa'\cos b+a^2)^{-\frac{1}{2}}$ be developed according to the cosines of the multiples of the arc b, it will have the form[36]

$$(a'^2-2aa'\cos b+a^2)^{-\frac{1}{2}}=\tfrac{1}{2}A_0+A_1.\cos b+A_2.\cos 2b+A_3.\cos 3b+\text{\&c.}$$

in which A_0, A_1, &c., are functions of a and a' alone; in fact if to abridge $\frac{a}{a'}=\alpha$, the binomial theorem gives

$$A_0=\frac{2}{a'}\left\{1+\left(\frac{1}{2}\right)^2\cdot\alpha^2+\left(\frac{1.3}{2.4}\right)^2\cdot\alpha^4+\left(\frac{1.3.5}{2.4.6}\right)\cdot\alpha^6+\text{\&c.}\right\},$$

the other coefficients are similar functions of the powers of α; but a general method of finding these coefficients in more convergent series will be given afterwards. Thus,

$$R' = m'\left\{\tfrac{1}{2}A_0 + \left(A_1 - \frac{a}{a'^2}\right)\cdot\cos b + A_2 .\cos 2b + \&c.\right\}$$

and if i represent every whole number either positive or negative including zero, the general term of this series is

$$R' = \frac{m'}{2}\cdot\Sigma .A_i .\cos ib\ ,$$

provided that when[37] $i = 1$, $\left(A_1 - \frac{a}{a'^2}\right)$ be put for A_1.

Again, if

$$\left(a'^2 - 2aa'\cos b + a^2\right)^{-\frac{3}{2}} = \tfrac{1}{2}B_0 + B_1 .\cos b + B_2 .\cos 2b + B_3 .\cos 3b + \&\text{c.}$$

its general term is

$$\frac{m'}{2}\cdot\Sigma .B_i .\cos ib\ ;$$

and as

$$\frac{dR'}{da} = \frac{m'}{2}\cdot\Sigma\cdot\left(\frac{dA_i}{da}\right)\cdot\cos ib;\quad \frac{dR'}{da'} = \frac{m'}{2}\cdot\Sigma\cdot\left(\frac{dA_i}{da'}\right)\cdot\cos ib;$$

$$\frac{dR'}{ndt} = -\frac{m'}{2}\cdot\Sigma\cdot iA_i\cdot\sin ib;\quad \frac{d^2R'}{da^2} = -\frac{m'}{2}\cdot\Sigma\cdot\left(\frac{d^2A_i}{da^2}\right)\cdot\cos ib;$$

&c. &c.

The development of R is

$$\begin{aligned} R = {} & \frac{m'}{2}\cdot\Sigma .A_i .\cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{2}\cdot u\cdot\Sigma\cdot a\cdot\left(\frac{dA_i}{da}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{2}\cdot u'\cdot\Sigma\cdot a'\cdot\left(\frac{dA_i}{da'}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & - \frac{m'}{2}\cdot\left(v' - v\right)\cdot\Sigma\cdot i\cdot A_i\cdot\sin i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{4}\cdot u^2\cdot\Sigma\cdot a^2\left(\frac{d^2A_i}{da^2}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \end{aligned}$$

$$+\frac{m'}{2}\cdot uu'\cdot\sum\cdot aa'\left(\frac{d^2A_i}{da\cdot da'}\right)\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\frac{m'}{4}\cdot u'^2\cdot\sum\cdot a'^2\left(\frac{d^2A_i}{da'^2}\right)\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{2}\cdot u\cdot\left(v'-v\right)\cdot\sum\cdot ia\left(\frac{dA_i}{da}\right)\cdot\sin i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{2}\cdot u'\cdot\left(v'-v\right)\cdot\sum\cdot ia'\left(\frac{dA_i}{da'}\right)\cdot\sin i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{4}\cdot\left(v'-v\right)^2\cdot\sum\cdot i^2A_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'\cdot zz'}{a'^3}+\frac{3m'\cdot a\cdot z'^2}{2\cdot a'^4}\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'\left(z'-z\right)^2}{4}\cdot\sum\cdot B_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\frac{3m'\cdot a\cdot z'^2}{4}\cdot\left(v'-v\right)\cdot\sum\cdot B_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\&c.\quad\&c.$$

a series that may be extended indefinitely.

447. If $v_/$ be the projection of *v*, by articles 398 and 401, $v_/$ and the curtate distance are[38]

$$r_/=r\left(1-\tfrac{1}{2}s^2+\tfrac{3}{8}s^4-\&c.\right),$$
$$v_/=v-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\,.\sin 4v+\&c.\right\}$$

or, if the values of *r* and *v*, in article 392, be substituted,

$$r_/=a\left(1+\tfrac{1}{2}e^2-e\cos\left(nt+\in-\varpi\right)+\&c.\right)\cdot\left(1-\tfrac{1}{2}s^2+\&c.\right),$$
$$v_/=nt+e+2e\sin\left(nt+\in-\varpi\right)+\&c.-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\sin 4v+\&c.\right\}.$$

Where *a* is half the greater axis of the orbit of *m*, *e* the eccentricity, ϖ the longitude of the perihelion, θ the longitude of the ascending node, f the inclination of the orbit of *m* on the fixed ecliptic at the epoch, and $nt+\in$ the mean longitude of *m*.

But

$$r_/=a\left(1+u\right),\quad v_/=nt+\in+v;$$

hence

$$u=-e\cos\left(nt+\in-\varpi\right)+\tfrac{1}{2}e^2\left(1-\cos 2\left(nt+\in-\varpi\right)\right)-\tfrac{1}{2}\tan^2 f\,.\sin^2\left(nt+\in\right),$$
$$v=2e\,.\sin\left(nt+\in-\varpi\right)+\tfrac{5}{4}e^2\cdot\sin 2\left(nt+\in-\varpi\right)-\tan^2\tfrac{1}{2}f\cdot\sin 2\left(nt+\in\right)$$

when the approximation only extends to the squares and products of the eccentricities and inclinations.

In the same manner,

$$u' = -e'\cos\left(n't+\in'-v'\right)+\tfrac{1}{2}e'^2\left(1-\cos 2\left(n't+\in'-v'\right)\right)-\tfrac{1}{2}\tan^2 f'\cdot\sin^2\left(n't+\in'\right),$$
$$v' = 2e'.\sin\left(n't+\in'-v'\right)+\tfrac{5}{4}e'^2\cdot\sin 2\left(n't+\in'-v'\right)-\tan^2\tfrac{1}{2}f'\cdot\sin 2\left(n't+\in'\right).$$

448. The substitution of these quantities will give the value of R in series, if the products of the sines and cosines be replaced by the cosines of the sums and differences of the arcs, observing that cosines of the forms

$$\cos\left\{i\left(n't-nt+\in'-\in\right)+n't-nt+\in'-\in-v+v'\right\},$$
$$\cos\left\{i\left(nn't-nt+\in'-\in\right)+n't-nt+\in'=\in-v+v'\right\}$$
$$\cos\left\{i\left(n't-nt+\in'-\in\right)+n't+nt+\in'-\in-2v\right\}$$

become

$$\cos\left\{i\left(n't-nt+\in'-\in\right)-v+v'\right\},$$
$$\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2e-2v\right\}$$

by the substitution of $i-1$ for *i*, and cosines of the form

$$\cos\left\{i\left(n't-nt+\in'-\in\right)-n't+nt-\in'+\in+v'-v\right\}$$

become

$$\cos\left\{i\left(n't-nt+\in'-\in\right)+v'-v\right\},$$

by the substitution of $i+1$ for *i*.

449. Attending to these circumstances, it will be found that

$$R=\frac{m'}{2}\cdot\sum A_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\frac{m'}{2}\cdot M_0\cdot e\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v\right\}$$
$$+\frac{m'}{2}\cdot M_1\cdot e'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v'\right\}$$
$$+\frac{m'}{2}\cdot N_0\cdot e^2\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-2v'\right\}$$
$$+\frac{m'}{2}\cdot N_1\cdot ee'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-v-v'\right\}$$
$$+\frac{m'}{2}\cdot N_2\cdot e'^2\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-2v'\right\}$$ **(118)**

$$+\frac{m'}{2}\cdot N_3\cdot\left(e^2+e'^2\right)\cdot\cos i\left(n't-nt+\epsilon'-\epsilon\right)$$
$$+\frac{m'}{2}\cdot N_4\cdot ee'\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+v-v'\right\}$$
$$+\frac{m'}{2}\cdot N_5\cdot ee'\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)-v+v'\right\}$$
$$-\frac{m'\cdot zz'}{a'^3}+\frac{3m'\cdot a\cdot z'^2}{2a'^4}\cdot\cos\left(n't-nt+\epsilon'-\epsilon\right)$$
$$-\frac{m'\left(z-z'\right)^2}{4}\cdot\sum B_i\cdot\cos\left(n't-nt+\epsilon'-\epsilon\right)$$
$$+\frac{m'}{4}\cdot Q_0\cdot e'^3\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-3v'\right\}$$
$$+\frac{m'}{4}\cdot Q_1\cdot e'^2\cdot e\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-2v'-v\right\}$$
$$+\frac{m'}{4}\cdot Q_2\cdot e'\cdot e^2\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-v'-2v\right\}$$
$$+\frac{m'}{4}\cdot Q_3\cdot e^3\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-3v'\right\}$$
$$+\frac{m'}{4}\cdot z'^2\cdot e'\cdot\sum\cdot B_i\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+n't+\epsilon'-v'\right\}$$
$$+\frac{m'}{4}\cdot z'^2\cdot e\cdot\sum\cdot B_i\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-v\right\}$$
$$+\&\text{c.}\quad\&\text{c.}$$

The coefficients being

$$M_0=-\left\{a\left(\frac{dA_i}{da}\right)+2iA_i\right\};$$
$$M_1=-a'\left(\frac{dA_{i-1}}{da'}\right)+2\left(i-1\right)A_{(i-1)};$$
$$N_0=\tfrac{1}{4}\left\{i\left(4i-5\right)\right\}A_i+2\left(2i-1\right)a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right);$$
$$N_1=-\tfrac{1}{2}\left\{4\left(i-1\right)^2A_{(i-1)}+2\left(i-1\right)a\left(\frac{dA_{(i-1)}}{da}\right)-2\left(i-1\right)a'\left(\frac{dA_{(i-1)}}{da'}\right)-aa'\left(\frac{d^2A_{(i-1)}}{da\cdot da'}\right)\right\};$$
$$N_2=\tfrac{1}{4}\left\{\left(i-2\right)\left(4i-3\right)A_{(i-2)}-2\left(2i-3\right)a'\left(\frac{dA_{(i-2)}}{da'}\right)+a'^2\left(\frac{d^2A_{(i-2)}}{da'^2}\right)\right\};$$
$$N_3=-\tfrac{1}{2}\left\{4i^2A_i-2a\left(\frac{dA_i}{da}\right)-a^2\left(\frac{d^2A_i}{da^2}\right)\right\};$$

$$N_4 = \tfrac{1}{2}\left\{4(i-1)^2 A_{(i-1)} - 2(i-1)a\left(\frac{dA_{(i-1)}}{da}\right) - 2(i-1)a'\left(\frac{dA_{(i-1)}}{da'}\right) + aa'\left(\frac{d^2 A_{(i-1)}}{da \cdot da'}\right)\right\};$$

$$N_5 = \tfrac{1}{2}\left\{4(i+1)^2 A_{(i+1)} + 2(i+1)a\left(\frac{dA_{(i+1)}}{da}\right) + 2(i+1)a'\left(\frac{dA_{(i+1)}}{da'}\right) + aa'\left(\frac{d^2 A_{(i+1)}}{da \cdot da'}\right)\right\};$$

&c. &c.

450. But $z = r_{/}s = r_{/}\tan f \sin(nt + \epsilon - q)$, by article 435, or substituting the values of $r_{/}$ and $v_{/}$ in article 447, and rejecting the product $e\tan f$, it becomes

$$z = a \cdot \tan f \sin(nt + \epsilon - q);$$

also

$$z' = a' \cdot \tan f' \sin(n't + \epsilon' - q'),$$

f and f' being the inclinations of the orbits of m and m' on the ecliptic. These values of z and z' are referred to the ecliptic at the epoch; but if the orbit of m at the epoch be assumed to be the fixed plane, $f = 0$, $\tan f' = g$, the mutual inclination of the orbits of m and m', then $\prod$ being the longitude of the ascending node of the orbit of m' on that of m,

$$z = 0,\ z' = a'g \sin(n't + \epsilon' - \prod),$$

consequently the terms of R depending on z' with regard to g^2, eg^2, and $e'g^2$, become[39]

$$+\frac{m'}{2} \cdot N_6 \cdot g^2 \cdot \cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\prod\},$$
$$+\frac{m'}{2} \cdot N_7 \cdot g^2 \cdot \cos\{i(n't - nt + \epsilon' - \epsilon)\},$$
$$+\frac{m'}{4} \cdot Q_4 \cdot g^2 e' \cdot \cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v' - 2\prod\},$$
$$+\frac{m'}{4} \cdot Q_5 \cdot g^2 e \cdot \cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v - 2\prod\}.$$

451. It appears from this series that the sum of the terms independent of the eccentricities and inclinations of the orbits, is

$$\frac{m'}{2} \cdot \sum A_i \cos i(n't - nt + \epsilon' - \epsilon),$$

which is the same as if the orbits were circular and in one plane.

The sum of the terms depending on the first powers of the eccentricities has the form

$$\frac{m'}{2}\sum M\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+K\right\}.$$

Those depending on the squares and products of the eccentricities and inclinations may be expressed by

$$\frac{m'}{2}\sum N\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in+L\right\}$$
$$+\frac{m'}{2}\sum N'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+L'\right\}.$$

Those depending on the third powers and products of these elements are

$$\frac{m'}{4}\sum Q\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in+U\right\}$$
$$+\frac{m'}{4}\sum\cdot Q'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+U'\right\},$$
&c. &c.

It may be observed that the coefficient of the sine or cosine of the angle v has always the eccentricity e for factor; the coefficient of the sine or cosine of $2v$ has e^2 for factor; the sine or cosine of $3v$ has e^3, and so on: also the coefficient of the sine or cosine of q has $\tan\cdot f$ for factor; the sine or cosine of $2q$ has $\tan^2.f$ for factor, &c. &c.

Determination of the Coefficients of the Series R

452. In order to complete the development of R, the coefficients A_i and B_i, and their differences, must be determined. Let

$$\left(a'^2-2aa'\cos b+a^2\right)^{-3}=A^{-3}=\tfrac{1}{2}A_0+A_1\cos b+A_2.\cos 2b+\&c.$$

The differential of which is

$$A^{-3-1}2saa'\sin b=A_1\sin b+2A_2\sin 2b+3A_3\sin 3b+\&c.$$

multiplying both sides of this equation by A, and substituting for A^{-3}, it becomes

$$2saa'\sin b\left\{\tfrac{1}{2}A_0+A_1\cos b+A_2\cos 2b+\&c.\right\}$$
$$=\left(a'^2-2aa'\cos b+a^2\right)\left\{A_1\sin b+2A_2\sin 2b+\&c.\right\}.$$

If it be observed that

$$\cos b \sin b = \tfrac{1}{2}\cos 2b, \text{ \&c.}$$

when the multiplication is accomplished, and the sines and cosines of the multiple arcs put for the products of the sines and cosines, the comparison of the coefficients of like cosines gives

$$A_2 = \frac{\left(a^2 + a'^2\right)A_1 - saa'A_0}{aa'(2-s)};$$
$$A_3 = \frac{2\left(a^2 + a'^2\right)A_2 - (1+s)\,a\,a'A_1}{aa'(3-s)}$$

and generally

$$A_i = \frac{(i-1)\left(a^2 + a'^2\right)A_{(i-1)} - (i+s-2)\,a\,a'A_{(i-2)}}{(i-s)\,aa'}; \qquad \textbf{(119)}$$

in which i may be any whole number positive or negative, with the exception of 0 and 1. Hence A_i will be known, if A_0, A_1 can be found.

Let

$$A^{-3-1} = \tfrac{1}{2}B_0 + B_1 \cos b + B_2 \cos 2b + \text{\&c.}$$

multiplying this by

$$\left(a^2 - 2aa'\cos b + a'^2\right),$$

and substituting the value of A^{-3} in series

$$\tfrac{1}{2}A_0 + A_1 \cos b + A_2 \cos 2b + \text{\&c.}$$
$$= \left(a^2 - 2aa'\cos b + a'^2\right)\left(\tfrac{1}{2}B_0 + B_1 \cos b + B_2 \cos 2b + \text{\&c.}\right)$$

the comparison of the coefficients of like cosines gives

$$A_i = \left(a^2 + a'^2\right).B_i - aa'.B_{(i-1)} - aa'.B_{(i+1)}.$$

But as relations must exist among the coefficients $B_{(i-1)}$, B_i, $B_{(i+1)}$, similar to those existing among $A_{(i-1)}$, A_i, $A_{(i+1)}$, the equation (119) gives, when $s+1$ and $i+1$ are put for *s* and *i*,

$$B_{(i+1)} = \frac{i\left(a^2 + a'^2\right)B_i - (i+s).\,a\,a'B_{(i-1)}}{(i-s)\,aa'}. \qquad \textbf{(120)}$$

If this quantity be put in the preceding value of A_i, it becomes

$$A_i = \frac{2saa' B_{(i-1)} - s\left(a^2 + a'^2\right) . B_i}{i-s}; \qquad \textbf{(121)}$$

or if $i+1$ be put for i,

$$A_{(i+1)} = \frac{2saa' B_i - s\left(a^2 + a'^2\right) . B_{(i+1)}}{i-s+1}; \qquad \textbf{(122)}$$

whence may be obtained, by the substitution of the preceding value of $B_{(i+1)}$,

$$A_{(i+1)} = \frac{s(i+s) \cdot aa'\left(a^2 + a'^2\right) B_{(i-1)} + s\left\{2(i-s)a^2a'^2 - i\left(a^2 + a'^2\right)\right\} B_i}{(i-s)(i-s+1) . aa'}.$$

If $B_{(i-1)}$ be eliminated between this equation and (121), there will result,

$$B_i = \frac{\frac{1}{s}(i+s)\left(a^2 + a'^2\right) \cdot A_i - \frac{2}{s}(i-s+1) \cdot aa' \cdot A_{(i+1)}}{\left(a'^2 - a^2\right)^2},$$

or substituting for $A_{(i+1)}$ its value given by equation (119),

$$B_i = \frac{\frac{1}{s}(s-1)\left(a^2 + a'^2\right) \cdot A_i + \frac{2}{s}(i+s-1) \cdot aa' \cdot A_{(i-1)}}{\left(a'^2 - a^2\right)^2}.$$

If to abridge $\frac{a}{a'} = \boldsymbol{a}$, the two last equations, as well as equation (119), when both the numerators and the denominators of their several members are divided by a'^2, take the form

$$A_i = \frac{(i-1)\left(1+\boldsymbol{a}^2\right) . A_{(i-1)} - (i+s-2) . \boldsymbol{a} \ . \ A_{(i-2)}}{(i-s)\boldsymbol{a}}, \qquad \textbf{(123)}$$

$$B_i = \frac{\frac{1}{s}(i+s)\left(1+\boldsymbol{a}^2\right) \cdot A_i - \frac{2}{s}(i-s+1) \cdot \boldsymbol{a}' \cdot A_{(i+1)}}{\left(1-\boldsymbol{a}^2\right)^2 a'^2}; \qquad \textbf{(124)}$$

$$B_i = \frac{\frac{1}{s}(s-i)(1+a^2)\cdot A_i + \frac{2}{s}(i+s-1)\cdot a'\cdot A_{(i-1)}}{(1-a^2)^2 a'^2}, \qquad \textbf{(125)}$$

which is very convenient for computation.

All the coefficients A_2, A_3, &c., B_0, B_1, &c., will be obtained from equations (123) and (125), when A_0, A_1, are known; it only remains, therefore, to determine these two quantities.

453.[40] Because

$$\cos b = \frac{c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}}{2},$$

c being the number whose hyperbolic logarithm is unity; therefore

$$a'^2 - 2aa'\cos b + a^2 = \left\{a' - ac^{b\sqrt{-1}}\right\}\cdot\left\{a' - ac^{-b\sqrt{-1}}\right\}$$

consequently,

$$A^{-3} = \left\{a' - ac^{b\sqrt{-1}}\right\}^{-3}\cdot\left\{a' - ac^{-b\sqrt{-1}}\right\}^{-3}.$$

But[41]

$$\left(a' - ac^{b\sqrt{-1}}\right)^{-3} = \frac{1}{a'^3}\left\{1 + sac^{b\sqrt{-1}} + \frac{s(s+1)}{2}a^2c^{2b\sqrt{-1}} + \&\text{c.}\right\},$$

$$\left(a' - ac^{-b\sqrt{-1}}\right)^{-3} = \frac{1}{a'^3}\left\{1 + sac^{-b\sqrt{-1}} + \frac{s(s+1)}{2}a^2c^{-2b\sqrt{-1}} + \&\text{c.}\right\};$$

the product of which is

$$\begin{aligned}A^{-3} = {} & \frac{1}{a'^{2s}}\left\{1 + s^2a^2 + \left(\frac{s(1+s)}{1.2}\right)^2 a^4 + \left(\frac{s(1+s)(2+s)}{1.2.3}\right)^2 a^6 + \&\text{c.}\right\} \\ & + \frac{2}{a'^{2s}}\left\{sa + \frac{s^2(1+s)}{1.2}a^3 + \frac{s(s+1)}{1.2}\cdot\frac{s(1+s)(2+s)}{1.2.3}a^5 + \&\text{c.}\right\}\times\left(c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}\right) + \&\text{c.}\end{aligned}$$

whence it appears that $c^{ib\sqrt{-1}}$, and $c^{-ib\sqrt{-1}}$ have always the same coefficients; and as

$$c^{ib\sqrt{-1}} + c^{-ib\sqrt{-1}} = 2\cos ib,$$

it is to see that this series is the same with

$$A^{-3} = \left(a'^2 - 2aa' \cos b + a^2\right)^{-3} = \tfrac{1}{2} A_0 + A_1 \cos b + \&c.$$

consequently,[42]

$$A_0 = \frac{2}{a'^{2s}} \left\{ 1 + s^2 a^2 + \left(\frac{s(1+s)}{1.2} \right)^2 a^4 + \left(\frac{s(1+s)(2+s)}{1.2.3} \right)^2 a^6 + \&c. \right\},$$
$$A_1 = \frac{2}{a'^{2s}} \left\{ sa + s \cdot \frac{s(1+s)}{1.2} \cdot a^3 + \frac{s(s+1)}{1.2} \cdot \frac{s(1+s)(2+s)}{1.2.3} a^5 + \&c. \right\}.$$

These series do not converge when $s = \tfrac{1}{2}$; but they converge rapidly when $s = -\tfrac{1}{2}$; then, however, A_0 and A_1 become the first and second coefficients of the development of

$$\left(a'^2 - 2aa' \cos b + a^2\right)^{\frac{1}{2}}.$$

Let[43] S and S' be the values of these two coefficients in this case, then

$$S = a' \left\{ 1 + \left(\frac{1}{2} \right)^2 a^2 + \left(\frac{1.1}{2.4} \right)^2 a^4 + \left(\frac{1.1.3}{2.4.6} \right)^2 a^6 + \&c. \right\}$$
$$S' = -a' \left\{ a - \frac{1.1}{2.4} a^3 - \frac{1.1.1.3}{4.2.4.6} a^6 - \frac{1.3.5.1.1.3.5.7}{4.6.8.2.4.6.8.10} a^7 \text{-\&c.} \right\}$$

and as the values of A_0, A_1, may be obtained in functions of S and S', the two last series form the basis of the whole computation.

Because[44] A_0, A_1, become S and S' when $s = -\tfrac{1}{2}$, and that B_i becomes A_i; if $s = -\tfrac{1}{2}$, and $i = 0$, equation (124) gives[45]

$$A_0 = \frac{4aS + 3\left(1 + a^2\right) S'}{\left(1 - a^2\right)^2 \cdot a'^2}.$$

and if $s = -\tfrac{1}{2}$, and $i = 1$, equation (125) gives

$$B_0 = \frac{\left(1 + a^2\right) A_0 - 2a A_1}{a'^2 \cdot \left(1 - a^2\right)^2};$$

and substituting the preceding values of A_0 and A_1, it becomes

$$B_0 = \frac{2S}{a'^4 \cdot \left(1 - a^2\right)^2} .$$

In the same manner it will be found that

$$B_1 = \frac{-3S'}{a'^4 \cdot \left(1 - a^2\right)^2} .$$

454. It now remains to determine the differences of A_i and B_i with regard to a. Resume

$$A^{-3} = \tfrac{1}{2} A_0 + A_1 \cos b + A_2 \cos 2b + \&\text{c.}$$

and take its differential with regard to a, observing that

$$\frac{dA}{da} = 2\left(a - a'\right)\cos b \ ;$$

then

$$-2s . \left(a - a'\right)\cos b \cdot A^{-3-1} = \tfrac{1}{2} \cdot \frac{dA_0}{da} + \frac{dA_1}{da} \cdot \cos b + \frac{dA_2}{da} \cdot \cos 2b + \&\text{c.}$$

But

$$A = a'^2 - 2aa' . \cos b + a^2$$

gives

$$a - a' \cos b = \frac{A + a^2 - a'^2}{2a};$$

therefore

$$A^{-3} + \left(a^2 - a'^2\right) A^{-3-1} = -\frac{1}{2} \cdot \frac{a}{s} \cdot \frac{dA_0}{da} - \frac{a}{s} \cdot \frac{dA_1}{da} \cos b - \frac{a}{s} \cdot \frac{dA_2}{da} \cos 2b - \&\text{c.}$$

or, substituting the values of A^{-3} and A^{-3-1} in series

$$\tfrac{1}{2} A_0 + A_1 \cos b + A_2 \cos 2b + \&\text{c.} + \left(a^2 - a'^2\right) \times \left\{\tfrac{1}{2} B_0 + B_1 \cos b + B_2 \cos 2b + \&c.\right\} =$$
$$-\frac{1}{2} \cdot \frac{a}{s} \cdot \frac{dA_0}{da} - \frac{a}{s} \cdot \frac{dA_1}{da} \cos b - \frac{a}{s} \cdot \frac{dA_2}{da} \cos 2b - \&\text{c.}$$

and the comparison of like cosines gives the general expression,

$$\frac{dA_i}{da} = \frac{s\left(a'^2 - a^2\right)}{a} \cdot B_i - \frac{s}{a} A_i ; \qquad \textbf{(126)}$$

or substituting B_i its value in (124),[46] it becomes[47]

$$\frac{dA_i}{da}=\left(\frac{ia'^2+(i+2s)\cdot a^2}{a\left(a'^2-a^2\right)}\right)A_i-\left(\frac{2(i-s+1)\cdot a'}{\left(a'^2-a^2\right)}\right)A_{(i+1)}.$$

If the differentials of this equation be taken with regard to a, and if, in the resulting equations, substitution be made for $\frac{dA_i}{da}$, $\frac{dA_{(i+1)}}{da}$ from the preceding formula, the successive differences of A_i, in functions of $A_{(i+1)}$, $A_{(i+2)}$, will be obtained.

Coefficients of the series R

455. If $\frac{1}{2}$ be put for s in the preceding equation, and in equation (123), and if it be observed that in the series R, article 446, $\frac{dA_i}{da}$ is always multiplied by a, $\frac{d^2A_i}{da^2}$ by a^2, and so on; then where i is successively made equal to 0, 1, 2, 3, &c. the coefficients and their differences are,

$$A_0=\frac{2\left(1+a^2\right)S+6aS'}{a'^2\left(1-a^2\right)^2}$$

$$A_1=\frac{4aS+3\left(1+a^2\right)S'}{a'^2\left(1-a^2\right)^2}$$

$$A_2=\frac{1}{3a}\cdot\left\{2\left(1+a^2\right)A_1-aA_0\right\}$$

$$A_3=\frac{1}{5a}\cdot\left\{4\left(1+a^2\right)A_2-3aA_1\right\}$$

$$A_4=\frac{1}{7a}\cdot\left\{6\left(1+a^2\right)A_3-5aA_2\right\}$$

$$A_5=\frac{1}{9a}\cdot\left\{8\left(1+a^2\right)A_4-7aA_3\right\}$$

&c. &c.

$$a\left(\frac{d.A_0}{da}\right)=\frac{1}{1-a^2}\left\{a^2A_0-aA_1\right\}$$

$$a\left(\frac{d.A_1}{da}\right)=\frac{1}{1-a^2}\left\{\left(1+2a^2\right)A_1-3aA_2\right\}$$

$$a\left(\frac{d.A_2}{da}\right)=\frac{1}{1-a^2}\left\{\left(2+3a^2\right)A_2-5aA_3\right\}$$

$$a\left(\frac{d.A_3}{da}\right)=\frac{1}{1-a^2}\left\{\left(3+4a^2\right)A_3-5aA_4\right\}$$

&c. &c.

$$a^2\left(\frac{d^2\,.\,A_0}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{2a^4A_0+\left(a-3a^3\right)A_1\right\}$$

$$a^2\left(\frac{d^2\,.\,A_1}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{\left(2-4a^2\right)A_1-\left(a-3a^3\right)A_0\right\}$$

$$a^2\left(\frac{d^2\,.\,A_2}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\begin{Bmatrix}\left\{\left(2+3a^2\right)^2+5a^2\left(1+a^2\right)-2\left(1-a^2\right)^2\right\}A_2\\ -5a\left(5+9a^2\right)A_3+5.7a^2A_4\end{Bmatrix}$$

$$a^2\left(\frac{d^2\,.\,A_3}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\begin{Bmatrix}\left\{\left(3+4a^2\right)^2+7a^2\left(1+a^2\right)-3\left(1-a^2\right)^2\right\}A_3\\ -7a\left(7+11a^2\right)A_4+7.9\cdot a^2A_5\end{Bmatrix}$$

$$a^2\left(\frac{d^2\,.\,A_4}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\begin{Bmatrix}\left\{\left(4+5a^2\right)^2+9a^2\left(1+a^2\right)-4\left(1-a^2\right)^2\right\}A_4\\ -9a\left(9+13a^2\right)A_5+9.11\cdot a^2A_6\end{Bmatrix}$$

&c. &c.

456. By the aid of equation (120), it is easy to see that

$$B_0=\frac{2S}{\left(a'^2-a^2\right)^2}$$

$$B_1=\frac{-3S'}{\left(a'^2-a^2\right)^2}$$

$$B_2=\frac{1}{a}\left\{2\left(1+a^2\right)B_1-3a\,B_0\right\}$$

$$B_3=\frac{1}{3a}\left\{4\left(1+a^2\right)B_2-5a\,B_1\right\}$$

$$B_4=\frac{1}{5a}\left\{6\left(1+a^2\right)B_3-7a\,B_2\right\}$$

&c. &c.

$$a\left(\frac{dB_0}{da}\right)=\frac{3a^2B_0+a\,B_1}{1-a^2}$$

$$a\left(\frac{dB_1}{da}\right)=\frac{3a\,B_0+\left(2a^2-1\right)B_1}{1-a^2}$$

&c. &c.

457. The coefficient A_i and its differences have a very simple form, when expressed in functions of B_i, for equations (121) and (126) give

$$A_0 = \left(a'^2 + a^2\right)B_0 - 2aa'B_1$$
$$A_1 = 2aa'B_0 - \left(a'^2 + a^2\right)B_1$$
$$A_2 = \frac{2aa'B_1 - \left(a'^2 + a^2\right)B_2}{3}$$
$$A_3 = \frac{2aa'B_2 - \left(a'^2 + a^2\right)B_3}{5}$$

&c. &c.

$$a\left(\frac{dA_0}{da}\right) = a'aB_1 - a^2B_0$$
$$a\left(\frac{dA_1}{da}\right) = a'^2B_1 - aa'B_0$$
$$a\left(\frac{dA_2}{da}\right) = \tfrac{1}{3}\left\{\left(2a'^2 - a^2\right)B_2 - aa'B_1\right\}$$
$$a\left(\frac{dA_3}{da}\right) = \tfrac{1}{5}\left\{\left(3a'^2 - 2a^2\right)B_3 - aa'B_2\right\}$$
$$a\left(\frac{dA_4}{da}\right) = \tfrac{1}{7}\left\{\left(4a'^2 - 3a^2\right)B_4 - aa'B_3\right\}$$

&c. &c.

$$a^2\left(\frac{d^2A_0}{da^2}\right) = 2a^2B_0 - a'aB_1$$
$$a^2\left(\frac{d^2A_1}{da^2}\right) = 3aa'B_0 - 2a'^2B_1$$

&c. &c.

458. The differences of A_i and B_i, with regard to a' are obtained from their differences with regard to a, for A_i being a homogeneous function of a and a' of the dimension -1,

$$a\left(\frac{dA_i}{da}\right) + a'\left(\frac{dA_i}{da'}\right) = -A_i\,;$$

as readily appears from

$$\left(a^2 - 2aa'\cos b + a'^2\right)^{-\frac{1}{2}},$$

therefore,

$$a'\left(\frac{dA_i}{da'}\right) = -A_i - a\left(\frac{dA_i}{da}\right)$$
$$a'\left(\frac{d^2A_i}{da\,.\,da'}\right) = -2\left(\frac{dA_i}{da}\right) - a\left(\frac{d^2A_i}{da^2}\right)$$

$$a'\left(\frac{d^2A_i}{da'^2}\right)=2A_i+4a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right)$$
&c. &c.

Likewise B_i being a homogeneous function of the dimension -3,

$$a'\left(\frac{dB_i}{da'}\right)+a\left(\frac{dB_i}{da}\right)=-3B_i.$$

459. By means of these, all the differences of A_i, B_i, with regard to a', may be eliminated from the series R, so that the coefficients of article 449 become[48]

$$M_0=-a\left(\frac{dA_i}{da}\right)-2iA_i$$

$$M_1=a\left(\frac{dA_{(i-1)}}{da}\right)+2(i-1)A_{(i-1)}$$

$$N_0=\tfrac{1}{4}\left\{i(4i-5)A_i+2(2i-1)a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right)\right\}$$

$$N_1=-\tfrac{1}{2}\left\{(2i-2)(2i-1)A_{(i-1)}+2(2i-1)a\left(\frac{dA_{(i-1)}}{da}\right)+a^2\left(\frac{d^2A_{(i-1)}}{da^2}\right)\right\}$$

$$N_2=\tfrac{1}{4}\left\{(4i^2-7i+2)A_{(i-2)}+2(2i-1)a\left(\frac{dA_{(i-2)}}{da}\right)+a^2\left(\frac{d^2A_{(i-2)}}{da^2}\right)\right\}$$

$$N_3=-\tfrac{1}{2}\left\{4i^2A_i-2a\left(\frac{dA_i}{da}\right)-a^2\left(\frac{d^2A_i}{da^2}\right)\right\}$$

$$N_4=\tfrac{1}{2}\left\{(2i-2)(2i-1)A_{(i-1)}-2a\left(\frac{dA_{(i-1)}}{da}\right)-a^2\left(\frac{d^2A_{(i-1)}}{da^2}\right)\right\}$$

$$N_5=\tfrac{1}{2}\left\{(2i+2)(2i+1)A_{(i+1)}-2a\left(\frac{dA_{(i+1)}}{da}\right)-a^2\left(\frac{d^2A_{(i+1)}}{da^2}\right)\right\}$$

$$N_6=\tfrac{1}{4}aa'\Sigma B_{(i-1)}$$

$$N_7=-\tfrac{1}{4}aa'\left(B_{(i-1)}+B_{(i+1)}\right)$$

&c. &c.

When $i=1$, $N_6=\tfrac{1}{4}aa'B_0-\tfrac{1}{2}\frac{a}{a'^2}$, and $\tfrac{1}{2}\frac{a}{a'^2}$ must be added to N_7.

460. The series represented by S and S' which are the bases of the computation, are numbers given by observation: for if the mean distance of the earth from the sun be assumed as the unit, the mean distances of the other planets determined by observation, may be expressed in functions of that unit, so that $a = \frac{a}{a'}$, the ratio of the mean distance of m to that of m' is a given number, and as the functions are symmetrical with regard to a and a', the denominator of $\frac{a}{a'}$ may always be so chosen is to make a less than unity, therefore if eleven or twelve of the first terms be taken and the rest omitted, the values of S and S' will be sufficiently exact; or, if their sum be found, considering them as geometrical series whose ratio is $1-a^2$, the values of S and S' will be exact to the sixth decimal, which is sufficient for all the planets and satellites. Thus A_i, B_i, their differences, and consequently the coefficients M_0, M_1, N_0, &c. of the series R are known numbers depending on the mean distances of the planets from the sun.

461. All the preceding quantities will answer for the perturbations of m' when troubled by m, with the exception of A_1, which becomes $A_1 - \frac{a}{a'}$; and when employed to determine the perturbations of Jupiter's satellites, the equatorial diameter of Jupiter, viewed at his mean distance from the sun, is assumed as the unit of distance, in functions of which the mean distances of the four satellites from the centre of Jupiter are expressed.

Notes

[1] See note 16, *Preliminary Dissertation*.

[2] This *reads* $a = \text{Func.}\left(x, y, z\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \cdot t\right)$ in the 1st edition.

[3] The differential element dt is missing in the middle term below in the 1st edition.

[4] See note 6, *Book I, Chapter II*.

[5] The third equation *reads* $d^2z = dt\left(\frac{dR}{dz}\right)$ in the 1st edition.

[6] Punctuation after the frst three equations is changed from commas to semicolons, and from a period to a semicolon after the sixth.

[7] The 1st edition uses a non-italicized d rather than the italicized d in the differential element $\mathrm{d}R$.

[8] The right side *reads* $\frac{1}{\sqrt{1+e^2}}\{1 - l(c^{(v-v)\sqrt{-1}} + c^{-(v-v)\sqrt{-1}}\} + l^2\left(c^{2(v-v)\sqrt{-1}} + c^{-2(v-v)\sqrt{-1}}\right) - \&c.\}$ in 1st edition.

[9] The equation *reads* $\int ndt + e = v + \mathrm{E}^{(1)}\sin(v - v) + \frac{1}{2}\mathrm{E}^{(2)}\sin 2(v - v) + \&c.\}$ in the 1st edition.

[10] The 1st edition reproduces only the first three terms of equation (101).

[11] The second right hand term in this expression *reads* $\left(\frac{dR}{dv'}\right)\frac{\sin v_/}{r_/}$ in the 1st edition.

[12] Punctuation added after first equation.

[13] Punctuation added after first equation.

[14] Punctuation added after 2nd expression.

15 The 1st edition uses a comma after the third equation.

16 The 2nd term in (110) contains a unbalanced bracket and *reads* $-a^2 ndt\sqrt{1-e^2}\cos(v-v\ )\}\left(\frac{dR}{dr}\right)$ in 1st edition.

17 The middle term below contains a comma in the denominator and *reads* $1+e\cos(v-v\ ),$ in the 1st edition.

18 The first equation in the 1st edition uses the undefined angle symbol j rather than f.

19 Multiplier symbol added to right hand side of second equation.
20 Punctuation added after the next three terms.
21 The closing bracket in the first term of the second equation is omitted in the 1st edition.
22 A printing error in the 1st edition uses the symbol j for f in the second term below.
23 These two equations are presented in the reverse order in following development.
24 a^2 *reads* a in the next two equations (published erratum).
25 A printing error in the 1st edition places the punctuation in the next two expressions inside the equations.
26 Punctuation added after expression for $d\in$ in equations (115) below.
27 See note 1.
28 Punctuation added after first definition.
29 Punctuation added after each expression.
30 Punctuation added after each expression.

31 The 2nd expression is obtained from the identity $\sin^2\Pi+\cos^2\Pi=1$.
32 m' *reads* m in the numerator of 1st term in the 1st edition (published erratum).
33 Product symbol inserted before sines in both expressions.
34 Misprints in the arguments of the cosines in the numerator of the second term and denominator of the fifth term are printed $\cos(v'_{/}-v)$ and $\cos(v'-v_{/})$ respectively in the 1st edition.

35 Note that the order of the terms a^2 and a'^2 is reversed in the previous equation.
36 Product symbol is inserted into the second and third terms.
37 Parentheses in 2nd term are omitted in the 1st edition.
38 The 2nd equation *reads* $v_{/}=v-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\cdot\sin 4v+\right\}$ &c. in the 1st edition.
39 The 2nd term *reads* $+\frac{m'}{2}\cdot N_7\cdot g^2\cdot\cos i\left(n't-nt+\in'-\in\right)$ in the 1st edition.
40 Article is numbered (454) in the 1st edition (published erratum).
41 Not capitalized in the 1st edition.
42 Punctuation added after second expression.
43 We have used the symbol *S* (also italicized) in another context to represent the solar mass elsewhere in this text.
44 A_1 *reads* $A_{/}$ in the 1st edition.

45 The numerator *reads* $4aS+3\left(1+a^2\right)S'$ in the 1st edition (published erratum).
46 Actually the value for B_i given before equation (123).
47 The numerator in the second term in the 1st edition contains a misprint in which a' *reads* a^1.
48 The third term in N_3 *reads* $-a^2\left(\frac{dA_i}{da^2}\right)$ in the 1st edition.