BOOK II

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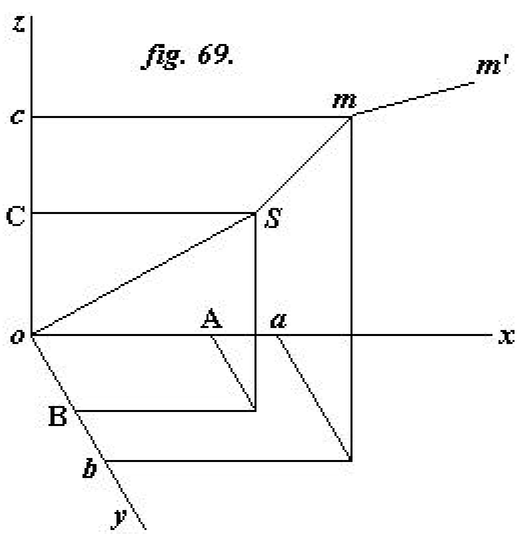
CHAPTER III

ON THE DIFFERENTIAL EQUATIONS OF THE MOTION OF A SYSTEM OF BODIES, SUBJECTED TO THEIR MUTUAL ATTRACTIONS

344. AS the earth which we inhabit is a part of the solar system, it is impossible for us to know any thing of its absolute motions; our observations must therefore be limited to its relative motions. In estimating the relative motion of planets, it is usual to refer them to the centre of the sun, and those of satellites to the centres of their primary planets. The sun and planets mutually attract each other; but in estimating the motions of a planet, the sun is supposed to be at rest, and all the motion is referred to the planet, which thus moves in consequence of the difference between its own action, and that of the sun. It is the same with regard to satellites and their primaries.

345. To determine the relative motions of a system of bodies $m,\ m',\ m''$, &c. fig. 69, considered as points revolving about one body S, which is the centre of their motions–



Let $\bar{x},\ \bar{y},\ \bar{z}$, be the co-ordinates of S referred to o as an origin, and $x,\ y,\ z,\ x',\ y',\ z'$, &c. the co-ordinates of the bodies $m,\ m'$, &c. referred to S as their origin. Then the co-ordinates of m when referred to o, are[1] $\bar{x}+x,\ \bar{y}+y,\ \bar{z}+z$, for it is easy to see that

$$\bar{x}+x=\mathrm{OA}+\mathrm{A}a,\quad \bar{y}+y=\mathrm{OB}+\mathrm{B}b,\quad \bar{z}+z=\mathrm{OC}+\mathrm{C}c\,.$$

In the same manner, the co-ordinates of m', when referred to o, are[2] $\bar{x}+x',\ \bar{y}+y',\ \bar{z}+z'$, and so for the other bodies. Let the distances of the bodies from S, or

$$Sm=\sqrt{x^2+y^2+z^2}\quad Sm'=\sqrt{x'^2+y'^2+z'^2}\,,\ \&\text{c.}$$

re represented by $r,\ r',\ r''$, &c. and the masses by $m,\ m'$, &c. and S. The equations of the motion of m will be first determined.

346. The whole action of the system relative to m consists three parts:

1. Of the action of S on m.
2. Of the action of all the bodies $m',\ m'',\ m'''$, &c. on m.
3. Of the action of all the bodies $m,\ m',\ m''$, &c. on S.

These will be determined separately [below].

i. The action of S on m is $-\frac{S}{r^2}$, that is directly as its mass, and inversely as the square of its distance. It has a negative sign, because the body S draws m towards the origin of the co-ordinates. This force when resolved in the direction ox is $-\frac{Sx}{r^3}$; for the force $-\frac{S}{r^2}$ is to its component force in ox, as Sm to Aa, that is as r to x.

ii. The distance of m' from m is

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

for x, y, z, x', y', z', being the co-ordinates of m and m' referred to S as their origin, the distance of these bodies from each other is the diagonal of a parallelopiped whose sides are $x'-x$, $y'-y$, $z'-z$. For the same reason, the distance of m'' from m is

$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}\text{, \&c.}$$

in order to abridge, let

$$l=\frac{m\cdot m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}+\frac{m\cdot m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}+\text{\&c.}$$

it is evident that[3]

$$\frac{1}{m}\left(\frac{dl}{dx}\right)=\frac{m'(x'-x)}{\left\{(x'-x)^2+(y'-y)^2+(z'-z)^2\right\}^{\frac{3}{2}}}+\frac{m''(x''-x)}{\left\{(x''-x)^2+(y''-y)^2+(z''-z)^2\right\}^{\frac{3}{2}}}+\&c.$$

is the sum of the actions of all the bodies m', m'', &c. on m when resolved in the direction ox. Hence the whole action of the system on m resolved in the axis[4] ox is

$$\frac{1}{m}\left(\frac{dl}{dx}\right)-\frac{Sx}{r^3};$$

but by the general theorem of motion

$$\frac{1}{m}\left(\frac{dl}{dx}\right)-\frac{Sx}{r^3}=\frac{d^2(\overline{x}+x)}{dt^2}, \qquad \textbf{(86)}$$

for $\bar{x}+x$ is the co-ordinate *oa*, or the distance of *m* from *o* in the direction *ox*.[5]

iii. The action of *m* on *S* is $\frac{m}{r^2}$, and its component force in *ox* is $\frac{mx}{r^3}$; likewise the actions of m', m'', &c. on *S*, when resolved in the same axes, are $\frac{m'x'}{r'^3}$, $\frac{m''x''}{r''^3}$, &c. hence the action of the system on *S* in the axis[6] *ox*, may be expressed by[7] $\Sigma\cdot\frac{mx}{r^3}$; but by the general theorem

$$\Sigma\cdot\frac{mx}{r^3}=\frac{d^2\bar{x}}{dt^2},$$

for the co-ordinates of *S* alone vary by this action. Now, if[8] $\Sigma\cdot\frac{mx}{r^3}$ be put for $\frac{d^2\bar{x}}{dt^2}$, in the equation (86) it becomes

$$0=\frac{d^2x}{dt^2}+\frac{Sx}{r^3}+\Sigma\cdot\frac{mx}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$

which is the whole action of the system relatively to *m*, when resolved in the direction *ox*, and because

$$\Sigma\cdot\frac{my}{r^3}=\frac{d^2\bar{y}}{dt^2},\qquad \Sigma\cdot\frac{mz}{r^3}=\frac{d^2\bar{z}}{dt^2};$$

the other two component forces are[9]

$$0=\frac{d^2y}{dt^2}+\frac{Sy}{r^3}+\Sigma\cdot\frac{my}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dy}\right),$$
$$0=\frac{d^2z}{dt^2}+\frac{Sz}{r^3}+\Sigma\cdot\frac{mz}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dz}\right).$$

The same equations will give the motions of m', m'', &c. round *S*, if m', x', y', z'; m'', x'', y'', z'', &c. be successively put for *m*, *x*, *y*, *z*, and *vice versâ*, and the equations

$$\frac{d^2\bar{x}}{dt^2}=\Sigma\cdot\frac{mx}{r^3},\quad \frac{d^2\bar{y}}{dt^2}=\Sigma\cdot\frac{my}{r^3},\quad \frac{d^2\bar{z}}{dt^2}=\Sigma\cdot\frac{mz}{r^3},$$

determine the motion of S.

347. These equations, however, may be put under a more convenient form for

$$\Sigma\cdot\frac{mx}{r^3}=\frac{mx}{r^3}+\frac{m'x'}{r'^3}+\&c.$$

and if $S+m$ the sum of the masses of the sun and of a planet, or of a planet and its satellite, be[10] represented by $m,$ the equation in x becomes

$$0=\frac{d^2x}{dt^2}+\frac{mx}{r^3}+\frac{m'x'}{r'^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right).$$

The part $\frac{d^2x}{dt^2}+\frac{mx}{r^3}$ relates only to the undisturbed elliptical motion of m round S; it is much greater than the remaining part

$$\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$

which contains all the disturbances to which the body m is subject from the action of the other bodies of the system. [The term] $-\frac{1}{m}\left(\frac{d\,l}{dx}\right)$ contains the direct action of the bodies $m',$ $m'',$ &c. on m; but m is also troubled indirectly by the action of these bodies on[11] S; this part is contained in $\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}$+&c.

By the latter action S is drawn to or from m; and by the former, m is drawn to or from S; in both cases altering the relative positions of S and m. Let

$$R=\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}-\frac{m'(x'x+y'y+z'z)}{r'^3}$$
$$+\frac{m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}-\frac{m''(x''x+y''y+z''z)}{r''^3}+\&\text{c.}$$

where it is easy to see that

$$-\frac{dR}{dx}=\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$
$$-\frac{dR}{dy}=\frac{m'y'}{r'^3}+\frac{m''y''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dy}\right),$$

$$-\frac{dR}{dz}=\frac{m'z'}{r'^3}+\frac{m''z''}{r''^3}+\&c.-\frac{1}{m}\left(\frac{d\,l}{dz}\right),$$

and therefore the preceding equations become[12]

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=\left(\frac{dR}{dx}\right),$$
$$\frac{d^2y}{dt^2}+\frac{my}{r^3}=\left(\frac{dR}{dy}\right), \quad \textbf{(87)}$$
$$\frac{d^2z}{dt^2}+\frac{mz}{r^3}=\left(\frac{dR}{dz}\right).$$

The whole motions of the planets and satellites are derived from these equations, for S may either be considered to be the sun, and $m,\ m',$ &c. planets; or S may be taken for a planet, and $m,\ m',$ &c. for its satellites.

If one planet only moved round the sun, its orbit would be a perfect ellipse, but by the attraction of the other planets, its elliptical motion is very much altered, and rendered extremely complicated.

348. It appears then, that the problem of planetary motion, in its most general sense, is the determination of the motion of a body when attracted by one body, and disturbed by any number of others. The only results that can be obtained from the preceding equations, which express this general problem, are the principle of areas and living forces; and that the motion of the centre of gravity is uniform, rectilinear, and in no way affected by the mutual action of the bodies. As these properties have been already proved to exist in a system of bodies mutually attracting each other, whatever the law of the force might be, provided that it could be expressed in functions of the distance; it evidently follows, that they must exist in the solar system, where the force is inversely as the square of the distance, which is only a particular case of the more general theorem. As no other results can be obtained from these general equations in the present state of analysis, the effects of one disturbing body is estimated at a time, but as this can be repeated for each body in the system, the disturbing action of all the planets on any one may be found.

349. The problem of planetary motion when so limited is, to determine, at any given time, the place of a body when attracted by one body and disturbed by another, the masses, distances, and positions of the bodies being given. This is the celebrated problem of three bodies; it is extremely complicated, and the most refined and laborious analysis is requisite to select among the infinite number of inequalities to which the planets are liable, those that are perceptible, and to assign their values. Although this problem has employed the greatest mathematicians from Newton to the present day, it can only be solved by approximation.

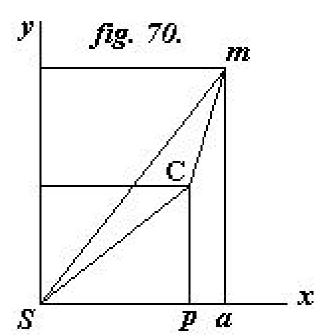
350. The action of a planet on the sun, or of a satellite on its primary, shortens its periodic time, if the planet be very large when compared with the sun, or the satellite when compared with its primary; for, as the ratio of the cube of the greater axis of the orbit to the square of the periodic time is proportional to the sum of the masses of the sun and the planet, Kepler's law would vary in the different orbits, according to the masses if they were considerable. But as the law is nearly the same for all the planets, their masses must be very small in comparison to that of the sun; and it is the same with regard to the satellites and their primaries. The volumes of the sun and planets confirm this; if the centre of the sun were to coincide with the centre of the earth, his volume would not only include the orbit of the moon, but would extend as far again, whence we may form some idea of his magnitude; and even Jupiter, the largest planet of the solar system, is incomparably smaller than the sun.

351. Thus any modifications in the periodic times, that could be produced by the action of the planets on the sun, must be insensible. As the masses of the planets are so small, their disturbing forces are very much less than the force of the sun, and therefore their orbits, although not strictly elliptical, are nearly so; and the areas described so nearly proportional to the time, that the action of the disturbing force may at first be neglected; then the body may be estimated to move in a perfect ellipse. Hence the first approximation is, to find the place of a body revolving round the sun in a perfect ellipse at a given time. In the second approximation, the greatest effects of the disturbing forces are found; in the third, the next greatest, and so on progressively, till they become so small, that they may be omitted in computation without sensible error. By these approximations, the place of a body may be found with very great accuracy, and that accuracy is verified by comparing its computed place with its observed place. The same method applies to the satellites.

Fortunately, the formation of the planetary system affords singular facilities for accomplishing these approximations: one of the principal circumstances is the division of the system into partial systems, formed by the planets and their satellites. These systems are such, that the distances of the satellites from their primaries are very much less than the distances of their primaries from the sun. Whence, the action of the sun being very nearly the same on the planet and on its satellites, the satellites move very nearly as if they were only influenced by the attraction of the planet.

Motion of the Centre of Gravity

352. From this formation it also follows, that the motion of the centre of gravity of a planet and its satellites, is very nearly the same as if all these bodies were united in one mass at that point.



Let C be the centre of gravity of a system of bodies m, m', m'', &c., as, for example, of a planet and its satellites, and let[13] S be any body not belonging to the system, as the sun.

It was shown, in the first book,[14] that the force which urges the centre of gravity of a system of bodies parallel to any straight line, Sx, is equal to the sum of the forces which urge the bodies m, m', &c. parallel to this straight line, multiplied respectively by their masses, the whole being divided by the sum of their masses.

It was also shown, that the mutual action and attraction of bodies united together in any manner whatever, has no effect on the centre of gravity of the system, whether at rest or in motion. It is, therefore, sufficient to determine the action of the body S, not belonging to the system, on its centre of gravity.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of C, fig. 70, the centre of gravity of the system referred to *S*, the centre of the sun; and let x, y, z, x', y', z', &c., be the co-ordinates of the bodies m, m', m'', &c., referred to C, their common centre of gravity. Imagine also, that the distances Cm, Cm', &c., of the bodies from their centre of gravity, are very small in comparison of *S*C, the distance of the centre of gravity from the sun. The action of the body m on the sun at *S*, when resolved in the direction *Sx*, is

$$\frac{m\cdot(\bar{x}+x)}{r^3},$$

in which m is the mass of the body, and [15]

$$r=\sqrt{(\bar{x}+x)^2+(\bar{y}+y)^2+(\bar{z}+z)^2}\,.$$

But the action of the sun on m is to the action of m on the sun, as *S*, the mass of the sun, to m, the mass of the body; hence the action of these two bodies on C, the centre of gravity of the system, is

$$-S\cdot\frac{\dfrac{m(\bar{x}+x)}{r^3}}{m}.$$

The same relation exists for each of the bodies; if we therefore represent the sum of the actions in the axis [16] [17] *ox* by

$$\Sigma\cdot\frac{m(\bar{x}+x)}{r^3},$$

and the sum of the masses $\Sigma.m$, by the whole force that acts on the centre of gravity in the direction *Sx* will be

$$-S\cdot\frac{\Sigma\cdot\dfrac{m(\bar{x}+x)}{r^3}}{\Sigma.m}.$$

Now, $\bar{x}+x$, fig. 70, is equal to $Sp+pa$, but *Sp* and *pa*, are the distances of the sun and of the body m from C, estimated on *Sx*; as *pa* is incomparably less than *Sp*, the square of *pa* may be omitted without sensible error, and also the squares of y and z, together with the products of these small quantities; then if

$$\bar{r}=S\mathrm{C}=\sqrt{\bar{x}^2+\bar{y}^2+z^2}\,,$$

the quantity $\frac{\bar{x}+x}{r^3}$ becomes[18]

$$\frac{\bar{x}+x}{\left\{\bar{r}^2+2\left(\bar{x}x+\bar{y}y+\bar{z}z\right)\right\}^{\frac{3}{2}}},$$

or

$$\left(\bar{x}+x\right)\left\{\bar{r}^2+2\left(\bar{x}x+\bar{y}y+\bar{z}z\right)\right\}^{-\frac{3}{2}}.$$

And expanding this by the binomial theorem, it becomes[19]

$$\frac{\bar{x}}{\bar{r}^3}+\frac{x}{\bar{r}^3}-\frac{3\bar{x}\left\{\bar{x}x+\bar{y}y+\bar{z}z\right\}}{\bar{r}^5}.$$

Now, the same expression will be found for x', y', z', &c., the co-ordinates of the other bodies; and as by the nature of the centre of gravity $\sum.mx=0$, $\sum.my=0$, $\sum.mz=0$, the expression[20]

$$-S\cdot\frac{\sum\cdot\frac{m\left(\bar{x}+x\right)}{r^3}}{\sum.m}\text{ becomes }-\frac{S\cdot\bar{x}}{r^3},$$

that is, when the squares and products of the small quantities x, y, z, &c., are omitted; hence the centre of gravity of the system is urged by the action of the sun in the direction Sx, as if all the masses were united in C, their common centre of gravity. It is evident that

$$-\frac{S\cdot\bar{y}}{\bar{r}^3},\qquad -\frac{S\cdot\bar{z}}{\bar{r}^3},$$

are the forces urging the centre of gravity in the other two axes.

353. In considering the relative motion of the centre of gravity of the system round S, it will be found that the action of the system of bodies m, m', m'', &c., on S in the axes ox, oy, oz, are

$$\frac{\bar{x}\,.\sum m}{\bar{r}^3};\ \frac{\bar{y}\,.\sum m}{\bar{r}^3};\ \frac{\bar{z}\,.\sum m}{\bar{r}^3},$$

when the squares and products of the distances of the bodies from their common centre of gravity are omitted. These act in a direction contrary to the origin. Whence the action of the system on S is nearly the same as if all their masses were united in their common centre of gravity; and the centre of gravity is urged in the direction of the axes by the sum of the forces, or by

$$-\{S+\sum .m\}\frac{\overline{x}}{\overline{r}^3},$$
$$-\{S+\sum .m\}\frac{\overline{y}}{\overline{r}^3}, \qquad \textbf{(88)}$$
$$-\{S+\sum .m\}\frac{\overline{z}}{\overline{r}^3};$$

and thus the centre of gravity moves as if all the masses $m,\ m',\ m''$, &c., were united in their common centre of gravity; since the coordinates of the bodies $m,\ m',\ m''$, &c., have vanished from all the preceding results, leaving only $\overline{x},\ \overline{y},\ \overline{z}$, those of the centre of gravity.

From the preceding investigation, it appears that the system of a planet and its satellites, acts on the other bodies of the system, nearly as if the planet and its satellites were united in their common centre of gravity; and this centre of gravity is attracted by the different bodies of the system, according to the same law, owing to the distance between planets being comparatively so much greater than that of satellites from their primaries.

Attraction of Spheroids

354. The heavenly bodies consist of an infinite number of particles subject to the law of gravitation; and the magnitude of these bodies bears so small a proportion to the distances between them, that they act on one another as if the mass of each were condensed in its centre of gravity. The planets and satellites are therefore considered as heavy points, placed in their respective centres of gravity. This approximation is rendered more exact by their form being nearly spherical: these bodies may be regarded as formed of spherical layers or shells, of a density varying from the centre to the surface, whatever the law may be of that variation. If the attraction of one of these layers, on a point interior or exterior to itself, can be found, the attraction of the whole spheroid may be determined.

Let C, fig. 71, be the centre of a spherical shell of homogeneous matter, and $\mathrm{CP}=a$, the distance of the attracted point P from the centre of the shell. As everything is symmetrical round CP, the whole attraction of the spheroid on P must be in the direction of this line. If *dm* be an element of the shell at m, and $f=m\mathrm{P}$ be its distance from the point attracted, then, assuming the action to be in the inverse ratio of the distance, $\frac{dm}{f^2}$ is the attraction of the particle on P; and if $\mathrm{CP}m=g$, this action, resolved in the direction CP, will be $\frac{dm}{f^2}\cdot\cos g$, and the whole attraction A of the shell on P, will be

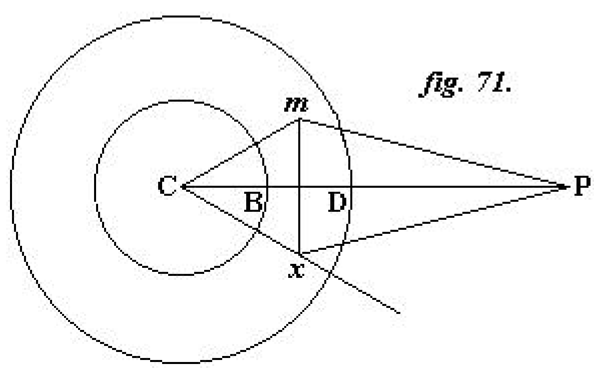


fig. 71.

$$A=\int\frac{dm\,.\cos g}{f^2}.$$

The position of the element *dm*, in space, will be determined by the angle $m\mathrm{CP}=q$, $\mathrm{C}m=r$, and by w, the inclination of the plane PC*m* on *m*C*x*. But, by article 278, $dm=r^2\sin q\, dr dv\, dq$; and from the triangle *CPm* it appears that

$$f^2=a^2-2ar\cos q+r^2;\ \cos g=\frac{a-r\cos q}{f};$$

hence

$$A=\int r^2\sin q\,.\,dr dv\, dq\cdot\frac{a-r\cos q}{f^3},$$

is the attraction of the whole shell on P, for the integral must be taken from $r=\mathrm{CB}$ to $r=\mathrm{CD}$, and from $q=0$, $w=0$ to $q=p$, $w=2p$, p being the semicircle whose radius is unity. The value of *f*, gives

$$\frac{df}{da}=\frac{a-r\cos q}{f};$$

hence

$$A=-\int r^2\cdot\sin q\, dr w dq\cdot\frac{d\dfrac{1}{f}}{da};$$

but as r, w, and q are independent of *a*,

$$A=-\frac{d\dfrac{\int r^2\sin q\,.\,dr w dq}{f}}{da}.$$

Thus the whole attraction of the spherical layer on the point P is obtained by taking the differential of

$$\int\frac{r^2\sin q\,.\,dr w\, dq}{f},$$

according to *a*, and dividing it by *da*. Let

$$\int\frac{r^2\sin q\,.\,dr w dq}{f}=V.$$

This integral from $w=0$ to $w=2p$, is

$$V=2p\int\frac{r^2 dr\,.\,dq\sin q}{f}.$$

But from the value of *f*, it is easy to find

$$\frac{dq\ \sin q}{f} = \frac{1}{ar}df\ ;$$

hence

$$V = \frac{2p}{a}\int rdr\,.\,df\ .$$

The integral with regard to q must be taken from $q = 0$ to $q = p$; but at these limits $f^2 = (a - r)^2$ and $f^2 = (a + r)^2$; and as f must always be positive, when the attracted point is within the spherical layer

$$f = r - a,\ \text{and}\ f = r + a\ ;$$

and when the attracted point P is without the spherical layer

$$f = a - r,\ \text{and}\ f = a + r\ ;$$

hence, in the first case,

$$V = 4p\int rdr$$

and in the second,

$$V = \frac{4p}{a}\int r^2 dr.$$

355. But the differential of V, according to a, and divided by da, when the sign is changed, is the whole attraction of the shell on P. Hence, from the first expression, $\frac{dV}{da} = 0$. Thus a particle of matter in the interior of a hollow sphere is equally attracted on all sides.

356. The second expression gives

$$-\frac{dV}{da} = \frac{4p}{a^2}\int r^2 dr.$$

The integral of this quantity from

$$r = \text{CB} = R' \text{ to } r = \text{CD} = R'',$$

is

$$-\frac{dV}{da} = \frac{4p}{3a^2}\left(R''^3 - R'^3\right),$$

which is the action of a spherical layer on a point without it.

If M be the mass of the layer whose thickness is $R'' - R'$, it will be equal to the difference of two spheres whose radii are R'' and R'; hence

$$M = \frac{4p}{3}\left(R''^3 - R'^3\right);$$

and therefore

$$A = \frac{M}{a^2}.$$

Thus the attraction of a spherical layer on a point exterior to it, is the same as if its whole mass were united in its centre.

357. If R', the radius of the interior surface, be zero, the shell will be changed into a sphere whose radius is R''. Hence the attraction of a homogeneous sphere on a point at its surface, or beyond it, is the same as if its mass were united at its centre.

These results would be the same were the attracting solid composed of layers of a density varying, according to any law whatever, from the centre to the surface; for, as they have been proved with regard to each of its layers, they must be true for the whole.

358. The celestial bodies then attract very nearly as if the mass of each was united in its centre of gravity, not only because they are far from one another, but because their forms are nearly spherical.

Notes

[1] An error in the 1st edition lists these three co-ordinates as $\bar{x} + x'$, $\bar{y} + y$, $\bar{z} + z$.

[2] An error in the 1st edition lists these three co-ordinates as $\bar{x} + x'$, $\bar{y} + y'$, $\bar{z} + z$.

[3] The right hand terms *read* $\dfrac{m'(x' - x)}{\left\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\right\}^{\frac{3}{2}}}$ in the 1st edition.

[4] This *reads* "axes" *for* "axis" in the 1st edition (published erratum).

[5] Punctuation added.

[6] This *reads* "axes" in the 1st edition (published erratum).

[7] This *reads* $\sum \frac{mx}{r^3}$ in 1st edition.

[8] This *reads* $\sum \frac{mx}{r^3}$ in 1st edition.

[9] The third terms in the two expressions below *read* $\sum \frac{my}{r^3}$, $\sum \frac{mz}{r^3}$ in the 1st edition.

[10] Misprint in 1st edition *reads* "by".

[11] Punctuation changed from a comma in 1st edition.

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[12] The second term in the last expression *reads* $\frac{mz}{\bar{r}^3}$ not $\frac{mx}{\bar{r}^3}$ as printed in the 1st edition (published erratum).

[13] The 1st edition is inconsistent in the use of italics for the body *S*. In this edition we use the italicized form throughout.

[14] See *Book I, Chapter IV*.

[15] The punctuation at the end of this expression is contained under the root in the 1st edition.

[16] This *reads* "axes", in the 1st edition.

[17] In fig. 70, the origin *o* must therefore coincide with *S*, although *o* is not labeled in the 1st edition figure.

[18] A bracket is missing in the following expression and *reads* $\frac{\bar{x}+x}{\left\{\bar{r}^2+2(\bar{x}x+\bar{y}y+\bar{z}z\right\}^{\frac{3}{2}}}$ in the 1st edition.

[19] The first term *reads* $\frac{}{\bar{r}^3}$ *for* $\frac{\bar{x}}{\bar{r}^3}$ as printed in the 1st edition (published erratum).

[20] Numerator below *reads* $\sum m$ in the 1st edition.