BOOK I

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CHAPTER VII

MOTION OF FLUIDS

General Equation of the Motion of Fluids

248. THE MASS of a fluid particle being $r\,dxdydz$, its momentum in the axis x arising from the accelerating forces is, by article 144,

$$\left\{X - \frac{d^2x}{dt^2}\right\} r\,dxdydz\,.$$

But the pressure resolved in the same direction is

$$\left(\frac{dp}{dx}\right) dxdydz\,.$$

Consequently the equation of the motion of a fluid mass in the axis ox, when free, is

$$\left\{X - \frac{d^2x}{dt^2}\right\} r - \frac{dp}{dx} = 0\,. \qquad \textbf{(56)}$$

In the same manner its motions in the axes y and z are[1]

$$\left\{X - \frac{d^2y}{dt^2}\right\} r - \frac{dp}{dy} = 0\,,$$
$$\left\{X - \frac{d^2z}{dt^2}\right\} r - \frac{dp}{dz} = 0\,. \qquad \textbf{(56)}$$

And by the principle of virtual velocities the general equation of fluids in motion is

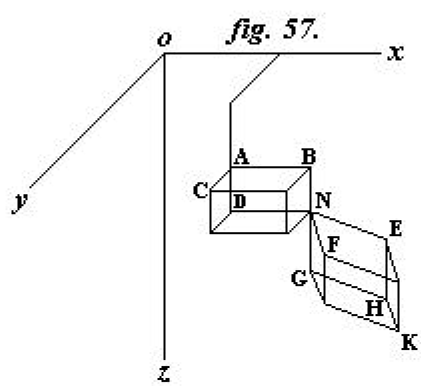
$$\{Xd\,x + Yd\,y + Zdz\} - \frac{d\,p}{r} = \frac{d^2x}{dt^2}d\,x + \frac{d^2y}{dt^2}d\,y + \frac{d^2z}{dt^2}d\,z\,. \qquad \textbf{(57)}$$

This equation is not rigorously true, because it is formed in the hypothesis of the pressures being equal on all sides of a particle in motion, which Poisson[2] has proved not to be the case; but, as far as concerns the following analysis, the effect of the inequality of pressure is insensible.

249. The preceding equation, however, does not express all the circumstances of the motion of a fluid. Another equation is requisite.

A solid always preserves the same form whatever its motion may be, which is by no means the case with fluids; for a mass ABCD, fig. 57, formed of particles possessing perfect mobility, changes its form by the action of the forces, so that it always continues to fit into the intervals of the surrounding molecules without leaving any void. In this consists the continuity of fluids, a property which furnishes the other equation necessary for the determination of their motions.

Equation of Continuity



250. Suppose at any given time the form of a very small fluid mass to be that of a rectangular parallelopiped ABCD, fig. 57. The action of the forces will change it into an oblique figure NEFK, during the indefinitely small time that it moves from its first to its second position. Now NEFG may differ from ABCD both in form and density, but not in mass; for if the density depends on the pressure, the same forces that change the form may also produce a change in the pressure, and, consequently, in the density; but it is evident that the mass must always remain the same, for the number of molecules in ABCD can neither be increased nor diminished by the action of the forces; hence the volume of ABCD into its primitive density must still be equal to [the] volume of NEFG into the new density; hence, if

$$r\,dxdydz\,,$$

be the mass of ABCD, the equation of continuity will be

$$d\;.\;r\;\;dx\;\;dy\;\;dz = 0\,. \qquad \textbf{(58)}$$

251. This equation, together with equations (56), will determine the four unknown quantities x, y, z, and p, in functions of the time, and consequently the motion of the fluid.

Development of the Equation of Continuity

252. The sides of the small parallelopiped, after the time dt, become

$$dx + d\;.\;dx,\;\; dy + d\;.\;dy,\;\; dz + d\;.\;dz\;;$$

or, observing that the variation of dx only arises from the increase of x, the co-ordinates y and z remaining the same, and that the variations of dy, dz, arise only from the similar increments of y and z; hence the edges of the new mass are

$$\mathrm{NE} = dx\left(1+\frac{d^2x}{dx}\right)$$
$$\mathrm{NG} = dy\left(1+\frac{d^2y}{dy}\right)$$
$$\mathrm{NF} = dz\left(1+\frac{d^2z}{dz}\right)$$

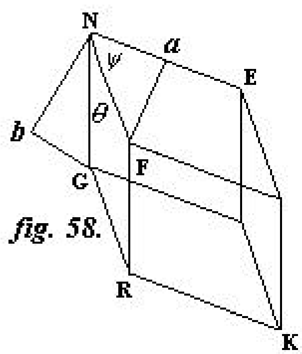


fig. 58.

If the angles GNF and FNE, fig. 58, be represented by q and y, the volume of the parallelopiped NK will be

$$\mathrm{NE}\,.\,\mathrm{NG}\sin q\,.\,\mathrm{NF}\sin y\;;$$

for

$$\mathrm{F}a = \mathrm{NF}\,.\sin y$$
$$\mathrm{N}b = \mathrm{NG}\,.\sin q\;,$$

Fa, Nb being at right angles to NE and RG; but as q and y were right angles in the primitive volume, they could only vary by indefinitely small arcs in the time dt; hence in the new volume

$$q = 90^\circ \pm dq,\; y = 90^\circ \pm dy\;,$$

consequently,

$$\sin q = \sin\left(90^\circ \pm dq\right) = \cos dq = 1 - \tfrac{1}{2}dq^2 + \&c.$$
$$\sin y = \sin\left(90^\circ \pm dy\right) = \cos dy = 1 - \tfrac{1}{2}dy^2 + \&c.$$

and omitting

$$dq^2,\;\; dy^2,\;\; \sin q = \sin y = 1\,,$$

and the volume becomes $\mathrm{NE}\,.\,\mathrm{NG}\,.\,\mathrm{NF}$; substituting for the three edges their preceding values, and omitting indefinitely small quantities of the fifth order, the volume after the time dt is

$$dxdydz\left\{1+\frac{d^2x}{dx}+\frac{d^2y}{dy}+\frac{d^2z}{dz}\right\}.$$

The density varies both with the time and space; hence r, the primitive density, is a function of t, x, y and z, and after the time dt, it is

$$r+\frac{d\,r}{dt}dt+\frac{d\,r}{dx}dx+\frac{d\,r}{dy}dy+\frac{d\,r}{dz}dz\;;$$

consequently, the mass, being the product of the volume and density, is, after the time dt, equal[3] to

$$dm=r\cdot dxdydz\left(1+\frac{d\,r}{dt}dt+\frac{d\,r}{dx}dx+\frac{d\,r}{dy}dy+\frac{d\,r}{dz}dz+r\,\frac{d^2x}{dx}+r\,\frac{d^2y}{dy}+r\,\frac{d^2z}{dz}\right).$$

And the equation

$$d\,.\,r\,.\,dxdydz=0$$

becomes

$$\frac{d\,r}{dt}+\frac{d\cdot r\,\frac{dx}{dt}}{dx}+\frac{d\cdot r\,\frac{dy}{dt}}{dy}+\frac{d\cdot r\,\frac{dz}{dt}}{dz}=0 \qquad \textbf{(59)}$$

as will readily appear by developing this quantity, which is the general equation of continuity.

253. The equations (56) and (59) determine the motions both of incompressible and elastic fluids.

254. When the fluid is incompressible, both the volume and density remain the same during the whole motion; therefore the increments of these quantities are zero; hence, with regard to the volume

$$\frac{d^2x}{dx}+\frac{d^2x}{dx}+\frac{d^2x}{dx}=0\;; \qquad \textbf{(60)}$$

and with regard to the density,

$$\frac{d\,r}{dt}+\frac{d\,r}{dx}dx+\frac{d\,r}{dy}d+\frac{d\,r}{dz}dz=0\,. \qquad \textbf{(61)}$$

255. By means of these two equations and the three equations (56), the five unknown quantities p, r, x, y, and z, may be determined in functions of t, which remains arbitrary; and therefore all the circumstances of the motion of the fluid mass may be known for any assumed time.

256. If the fluid be both incompressible and homogeneous, the density is constant, therefore $d\,r=0$, and as the last equation becomes identical, the motion of the fluid is obtained from the other four.

Second form of the Equation of the Motions of Fluids

257. It is occasionally more convenient to regard x, y, z, the co-ordinates of the fluid particle dm, as known quantities, and

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt},$$

its velocities in the direction of the co-ordinates, as unknown. In order to transform the equations (56) and (59) to suit this case, let

$$s=\frac{dx}{dt},\ u=\frac{dy}{dt},\ v=\frac{dz}{dt};$$

these quantities being functions of x, y, z, and t. The differentials of these equations when x, y, z, and t vary all at once; and when sdt, udt, vdt, are put for dx, dy, dz, become

$$\begin{aligned} ds &= \frac{ds}{dt}dt+\frac{ds}{dx}\cdot sdt+\frac{ds}{dy}\cdot udt+\frac{ds}{dz}\cdot vdt\,, \\ du &= \frac{du}{dt}dt+\frac{du}{dx}\cdot sdt+\frac{du}{dy}\cdot udt+\frac{du}{dz}\cdot vdt\,, \\ dv &= \frac{dv}{dt}dt+\frac{dv}{dx}\cdot sdt+\frac{dv}{dy}\cdot udt+\frac{dv}{dz}\cdot vdt\,. \end{aligned} \qquad \textbf{(62)}$$

And as

$$ds=\frac{d^2x}{dt},\ du=\frac{d^2y}{dt},\ dv=\frac{d^2z}{dt},$$

the equations (56) become, by the substitution of the preceding quantities,

$$\begin{aligned} \frac{dp}{dx} &= r\left\{\mathrm{X}-\frac{ds}{dt}-\frac{ds}{dx}\cdot s-\frac{ds}{dy}\cdot u-\frac{ds}{dz}\cdot v\right\} \\ \frac{dp}{dy} &= r\left\{\mathrm{Y}-\frac{du}{dt}-\frac{du}{dx}\cdot s-\frac{du}{dy}\cdot u-\frac{du}{dz}\cdot v\right\} \\ \frac{dp}{dz} &= r\left\{\mathrm{Z}-\frac{dv}{dt}-\frac{dv}{dx}\cdot s-\frac{dv}{dy}\cdot u-\frac{dv}{dz}\cdot v\right\} \end{aligned} \qquad \textbf{(63)}$$

and by the same substitution, the equation (59) of continuity becomes

$$\frac{dr}{dt}+\frac{d\,.\,rs}{dx}+\frac{d\,.\,ru}{dy}+\frac{d\,.\,rv}{dz}=0\,, \qquad \textbf{(64)}$$

which, for incompressible and homogeneous fluids, is

$$\frac{ds}{dx}+\frac{du}{dy}+\frac{dv}{dz}=0. \qquad \textbf{(65)}$$

The equations (63) and (64) will determine s, u, and v, in functions of x, y, z, t, and then the equations

$$dx = sdt \quad dy = udt \quad dz = vdt$$

will give x, y, z, in functions of the time. The whole circumstances of the fluid mass will therefore be known.

Integration of the Equations of the Motions of Fluids

258. The great difficulty in the theory of the motion of fluids, consists in the integration of the partial equations (63) and (64), which has not yet been surmounted, even in the most simple problems. It way, however, be effected in a very extensive case, in which

$$sdx + udy + vdx$$

is a complete differential of a function f, of the three variable quantities x, y, z; so that

$$sdx + udy + vdz = df\ .$$

259. If in the equation (57) the variations which are arbitrary, be made equal to the differentials of the same quantities; and if, as in nature, the accelerating forces X, Y, Z be functions of the distance, then $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ will be a complete differential, and may be expressed by dV, so that the equation in question becomes

$$\frac{dp}{r}=d\mathrm{V}-dx\cdot\frac{d^2x}{dt^2}-dy\cdot\frac{d^2y}{dt^2}-dz\cdot\frac{d^2z}{dt^2} \qquad \textbf{(66)}$$

But the function f gives the velocities of the fluid mass in the directions of the axes, *viz*.

$$s=\frac{df}{dx},\ u=\frac{df}{dy},\ v=\frac{df}{dz}.$$

By the substitution of these values in equation (62), ds, du, dv, and consequently

$$\frac{d^2x}{dt},\quad \frac{d^2y}{dt},\quad \frac{d^2z}{dt},$$

will be obtained in functions of f, by which the preceding equation becomes[4]

$$\frac{dp}{r} = d\mathrm{V} - \frac{ds}{dt}\cdot dx - \frac{du}{dt}\cdot dy - \frac{dv}{dt}\cdot dz - \tfrac{1}{2} d\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2} \right).$$

Now

$$\frac{ds}{dt}\cdot dx + \frac{du}{dt}\cdot dy + \frac{dv}{dt}\cdot dz = d\cdot\frac{df}{dt};$$

consequently,[5]

$$\int \frac{dp}{r} = \mathrm{V} - \frac{df}{dt} - \tfrac{1}{2}\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2} \right). \qquad \textbf{(67)}$$

The constant quantity introduced by integration is included in the function f. By the same substitution, the equation of continuity becomes

$$\frac{dr}{dt} + \frac{d\,.\,r\dfrac{df}{dx}}{dx} + \frac{d\cdot r\dfrac{df}{dy}}{dy} + \frac{d\,.\,r\dfrac{df}{dz}}{dz} = 0\,. \qquad \textbf{(68)}$$

The two last equations determine the motion of the fluid mass in the case under consideration.

260. It is impossible to know all the cases in which the function $sdx+udy+vdz$ is an exact differential, but it may be proved that if it be so at any one instant, it will be an exact differential during the whole motion of a fluid.

Demonstration. Suppose that at any one instant it is a complete differential, it will then be integrable, and may be expressed by df; in the following instant it will become[6]

$$df + \frac{ds}{dt}\cdot dx + \frac{du}{dt}\cdot dy + \frac{dv}{dt}\cdot dz\,.$$

It will still be an exact differential, if

$$\frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz \text{ be one.}$$

Now the latter quantity being equal to $d\cdot\dfrac{df}{dt}$, equation (67) gives[7]

$$\frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz = d\mathrm{V} - \frac{dp}{r} - \tfrac{1}{2}d\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2} \right).$$

And if the density r be a function of p the pressure, the second member of this equation will be an exact differential, consequently the first member will be one also, and thus the function *sdx+udy+vdz* is a complete differential in the second instant, if it be one in the first; it will therefore be a complete differential during the whole motion of the fluid.

Theory of small Undulations of Fluids

261. If the oscillations of a fluid be very small, the squares and products of the velocities *s, u, v*, may be neglected: then the preceding equation becomes

$$d\mathrm{V} - \frac{dp}{r} = \frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz\ .$$

If r be a function of p, the first member will be a complete differential, therefore the second member, and consequently $sdx+udy+vdz$ is one also, so that the equation is capable of integration; and as its last member is equal to $d\cdot\frac{df}{dt}$, the integral is

$$\mathrm{V} - \int\frac{dp}{r} = \frac{df}{dt}\,. \qquad \textbf{(69)}$$

This equation, together with equation (68) of continuity, contain the whole theory of the small undulations of fluids.

262. An idea may be formed of these undulations by the effect of a stone dropped into still water; a series of small concentric circular waves will appear, extending from the point where the stone fell. If another stone be let fall very near the point where the first fell, a second series of concentric circular waves will be produced; but when the two series of undulations meet, they will cross, each series continuing its course independently of the other, the circles cutting each other in opposite points. An infinite number of such undulations may exist without disturbing the progress of one another. In sound, which is occasioned by undulations in the air, a similar effect is produced: in a chorus, the melody of one voice may be distinguished from the general harmony. Coexisting vibrations may also be excited in solid bodies, each undulation having its perfect effect, independently of the others. If the directions of the undulations coincide, their joint motions will be the sum or the difference of the separate motions, according as similar or dissimilar parts of the undulations are coincident. In undulations of equal frequency, when two series exactly coincide in point of time, the united velocity of the particular motions will be the greatest or least;–and if the undulations are of equal strength, they will totally destroy each other, when the time of the greatest direct motion of one undulation coincides with that of the greatest retrograde motion of the other.

The general principle of Interferences was first shown by Dr. Young[8] to be applicable to all vibratory motions, which he illustrated beautifully by the remarkable phenomena of two rays of light producing darkness, and the concurrence of two musical sounds producing silence. The first may be seen by looking at the flame of a candle through two extremely narrow parallel slits in a card; and the latter is rendered evident by what are termed beats in music.

The same principle serves to explain why neither flood nor ebb tides take place at Batsham in Tonquin on the day following the moon's passage across the equator; the flood tide arrives by one channel at the same instant that the ebb arrives by another, so that the interfering waves destroy each other.

Co-existing vibrations show the comprehensive nature and elegance of analytical formulae. The general equation of small undulations is the sum of an infinite number of equations, each of which gives a single series of undulations, like the surface of water in a shower, which at once contains an infinite number of undulations, and yet exhibits each independently of the rest.

Rotation of a Homogeneous Fluid

263. If a fluid mass rotates uniformly about an axis, its component velocity in the axis of rotation is zero; the velocities in the other two axes are angular velocities–independent of the time, the motion being uniform: indeed, the motion is the same with that of a solid body revolving about a fixed axis. If the mass revolves about the axis z, and if w be the angular velocity at the distance of unity from that axis, the component velocities will be

$$s = -wy, \quad u = wx, \quad v = 0;$$

and from equations (63) it will be easily found that

$$\frac{dp}{r} = d\mathrm{V} + w^2\left(xdx + ydy\right);$$

and if r be constant, the integral is

$$\frac{p}{r} = \mathrm{V} + \frac{w^2}{2}\left(x^2 + y^2\right).$$

The equation (65) of continuity will be satisfied, since

$$\frac{ds}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{dv}{dz} = 0.$$

264. This motion of a fluid mass is therefore possible, although it is a case in which the function $sdx+udy+vdz$ is not an exact differential; for by the substitution of the preceding values of the velocities, it becomes

$$sdx + udy + vdz = w\left(xdy - ydx\right),$$

an expression that cannot be integrated. Therefore, in the theory of the tides caused by the disturbing action of the sun and moon on the ocean, the function $sdx+udy+vdz$ must not be regarded as an exact differential, since it cannot be integrated even when there is no disturbance in its rotatory motion.

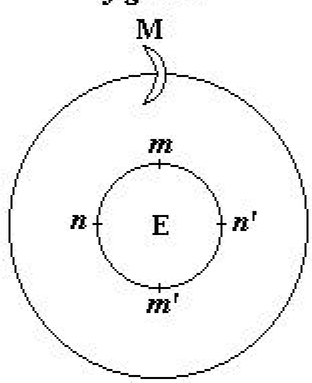
265. Thus a fluid mass or a fluid covering a solid of any form whatever, will rotate about an axis without alteration in the relative position of its particles. This would be the state of the ocean were the earth a solitary body, moving in space; but the attractions of the sun and moon not only trouble the ocean, but also cause commotions in the atmosphere, indicated by the periodic variations in the heights of the mercury in the barometer. From the vast distance of the sun and moon, their action upon the fluid particles of the ocean and atmosphere, is very small in comparison of that produced by the velocity of the earth's rotation, and by its attraction.

Determination of the Oscillations of a Homogeneous Fluid covering a Spheroid, the whole in rotation about an axis; supposing the fluid to be slightly deranged from its state of equilibrium by the action of very small forces

266. If the earth be supposed to rotate about its axis, uninfluenced by foreign forces, the fluids on its surface would assume a spheroidal form, from the centrifugal force induced by rotation; and a particle in the interior of the fluid would be subject to the action of gravitation and the pressure of the surrounding fluid only. But although the fluids would be moving with great velocity, yet to us they would seem at rest. When in this state the atmosphere and ocean are said to be in equilibrio.

Action of the Sun and Moon

fig. 59.



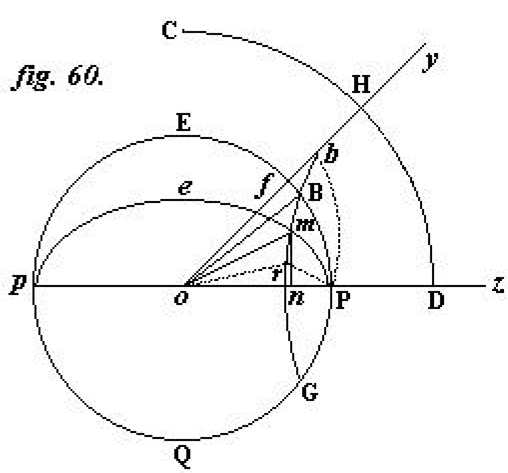
267. The action of the sun and moon troubles this equilibrium, and occasions tides in both fluids. The whole of this theory is perfectly general, but for the sake of illustration it will be considered with regard to the ocean. If the moon attracted the centre of gravity of the earth and all its particles with equal and parallel forces, the whole system of the earth and the waters that cover it, would yield to these forces with a common motion, and the equilibrium of the seas would remain undisturbed. The difference of the intensity and direction of the forces alone, trouble the equilibrium; for, since the attraction of the moon is inversely as the square of the distance, a molecule at m, under the moon M, is as much more attracted than the centre of gravity of the earth, as the square of EM is greater than the square of mM: hence the particle has a tendency to leave the earth, but is retained by gravitation, which this tendency diminishes. Twelve hours after, the particle is brought to m' by the rotation of the earth, and is then in opposition to the moon, which attracts it more feebly than it attracts the centre of the earth, in the ratio of the square of EM to the square of $m'\mathrm{M}$. The surface of the earth has then a tendency to leave the particle, but the gravitation of the particle retains it; and gravitation is also in this case diminished by the action of the moon. Hence, when the particle is at m, the moon draws the particle from the earth; and when it is at m', it draws the earth from the particle: in both instances producing an elevation of the particle above the surface of equilibrium of nearly the same height, for the diminution of the gravitation in each position is almost the same on account of the distance of the moon being great in comparison of the radius of the earth. The action of the moon on a particle at *n*, 90° distant from *m*, may be resolved into two forces–one in the direction of the radius *n*E, and the other tangent to the surface. The latter force alone attracts the particle towards the moon, and makes it slide along the surface; so that there is a depression of the water in *n* and n', at the same time that it is high water at *m* and m'. It is evident that, after half a day, the particle, when at n', will be acted on by the same force it experienced at *n*.

268. Were the earth entirely covered by the sea, the water thus attracted by the moon would assume the form of an oblong spheroid, whose greater axis would point towards the moon; since the column of water under the moon, and the direction diametrically opposite to her, would be rendered lighter in consequence of the diminution of their gravitation: and in order to preserve the equilibrium, the axis 90° distant would be shortened. The elevation, on account of the smaller space to which it is confined, is twice as great as the depression, because the contents of the spheroid always remain the same. If the waters were capable of instantly assuming the form of a spheroid, its summit would always be directed towards the moon, notwithstanding the earth's rotation; but on account of their resistance, the rapid motion of rotation prevents them from assuming at every instant the form which the equilibrium of the forces acting on them requires, so that they are constantly approaching to, and receding from that figure, which is therefore called the *momentary equilibrium* of the fluid. It is evident that the action, and consequently the position of the sun modifies these circumstances, but the action of that body is incomparably less than that of the moon.

Determination of the general Equation of the Oscillations of all parts of the Fluids covering the Earth

269. Let *p*EPQ, fig. 60, be the terrestrial spheroid, E*o* the equatorial radius, P*p* the axis of rotation. Suppose the spheroid to be entirely covered with the fluid–the ocean, for example; and let *pe*P, or *p*EP, represent the bottom of the sea, CD its surface, PD its depth; also let *o* be the centre of the spheroid and origin of the co-ordinates, and *om* the radius. Imagine *m* to be a fluid particle at any point below the surface of the fluid–at the bottom, for example. It is evident that this particle, moved by rotation alone, would be carried to B without changing its distance from the centre of the spheroid, or from the axis of rotation; so that the arcs P*m*, PB, are equal to each other, as also the radii *om*, *o*B. If *pe*P be assumed as a given meridian, the origin of the time, and g the first point in Aries,[9] then gPB is the longitude of the particle when arrived at B, and E*o*B is its latitude.

fig. 60.



Now, if the disturbing forces were to act on the particle during its rotation from *m* to B, they would cause it to move to *b*, some point not far from B. By the disturbing forces alone, the longitude of the particle at B would be increased by the very small angle BP*b*; the latitude would be diminished by the very small angle B*ob*, and its distance from the centre of the spheroid increased by *fb*. The angle g PB is the rotation of the earth, and any may be represented by $nt + v$, since it is proportional to the time, (by Article 213); but in the time *t*, the disturbing forces bring the particle to *b*: therefore the angle $nt + v$ must be increased by BP*b* or v. Hence

$$g\,\mathrm{P}b = nt + v \ + v\,.$$

Again, if q be the component of the latitude E*o*B, and *u*, its very small increment, B*ob*, the angle

$$\mathrm{P}o\mathrm{B} = q + u\,.$$

In the same manner, if *s* be the increment of the radius *r*, then

$$ob = r + s\,.$$

Hence the co-ordinates of the particle at *b* are,

$$x = (r+s)\cos(q + u)\,,$$
$$y = (r+s)\sin(q+u)\cos(nt + v \ + v)\,,$$
$$z = (r+s)\sin(q+u)\sin(nt + v \ + v)\,.$$

270. *v* and *u* very nearly represent the motion of the particle in longitude and latitude estimated from the terrestrial meridian P*ep*. These are so small, compared with *nt* the rotatory motion of the earth, that their squares may be omitted. But although the lateral motions *v*, *u* of the particle be very small, they are much greater than *s*, the increase in the length of the radius.

271. If these values of *x*, *y*, *z*, be substituted in (57) the general equation of the motion of fluids; and if to abridge

$$\mathrm{X}d\,x + \mathrm{Y}d\,y + \mathrm{Z}d\,z = dV\,,$$

then[10]

$$\begin{aligned} & r^2 dq\left\{\frac{d^2u}{dt^2} - 2n\sin q\,\cos q\left(\frac{dv}{dt}\right)\right\} \\ & + r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\,\cos q\left(\frac{du}{dt}\right) + \frac{2n\sin^2 q}{r}\left(\frac{ds}{dt}\right)\right\} \\ & + dr\left\{\left(\frac{d^2s}{dt^2}\right) - 2nr\sin^2 q\left(\frac{dv}{dt}\right)\right\} \\ & = \frac{n^2}{2}d\left\{(r+s)\sin(q+u)\right\}^2 + dV - \frac{dp}{r}\,, \end{aligned} \qquad \textbf{(70)}$$

will determine the oscillations of a particle in the interior of the fluid when troubled by the action of the sun and moon. This equation, however, requires modification for a particle at the surface.

Equation at the Surface

272. If DH, fig. 60, be the surface of the sea undisturbed in its rotation, the particle at B will only be affected by gravitation and the pressure of the surrounding fluid; but when by the action of the sun and moon the tide rises to *y*, and the particle under consideration is brought to *b*, the forces which there act upon it will be gravitation, the pressure of the surrounding fluid, the action of the sun and moon, and the pressure of the small column of water between H and *y*. But the gravitation acting on the particle at *b* is also different from that which affects it when at B, for *b* being farther from the centre of gravity of the system of the earth and its fluids, the gravity of the particle at *b* must be less than at B, consequently the centrifugal force must be greater: the direction of gravitation is also different at the points B and *b*.

273. In order to obtain an equation for the motion of a particle at the surface of the fluid, suppose it to be in a state of momentary equilibrium, then as the differentials of v, u, and s, express the oscillations of the fluid about that state, they must be zero, which reduces the preceding equation to

$$\frac{n^2}{2}d\left\{(r+s)\sin(q+u)\right\}^2+(dV)=0\,; \qquad \textbf{(71)}$$

for as the pressure at the surface is zero, $dp=0$, and (dV) represents the value of dV corresponding to that state. Thus in a state of momentary equilibrium, the forces (dV), and the centrifugal force balance each other.

274. Now dV is the sum of all the forces acting on the particle when troubled in its rotation into the elements of their directions, it must therefore be equal to (dV), the same sum suited to a state of momentary equilibrium, together with those forces which urge the particle when it oscillates about that state, into the elements of their directions. But these are evidently the variation in the weight of the little column of water H*y*, and a quantity which may be represented by dV', depending on the difference in the direction and intensity of gravity at the two points B and *b*, caused by the change in the situation of the attracting mass in the state of motion, and by the attraction of the sun and moon.

275. The force of gravity at *y* is so nearly the same with that at the surface of the earth, that the difference may be neglected; and if *y* be the height of the little column of fluid H*y*, its weight will be *gy* when the sea is in a state of momentary equilibrium; when it oscillates about that state; the variation in its weight will be $gd\,y$, g being the force of gravity; but as the pressure of this small column is directed towards the origin of the co-ordinates and tends to lessen them, it must have a negative sign. Hence in a state of motion,

$$dV=(dV)+dV'-gd\,y\,,$$

whence

$$(dV) = dV - dV' + gd\,y\,.$$

276. When the fluid is in momentary equilibrio, the centrifugal force is

$$\frac{n^2}{2}\{(r+s)\sin(q+u)\}^2;$$

but it must vary with $d\,y$, the elevation of the particle above the surface of momentary equilibrium. The direction Hy does not coincide with that of the terrestrial radius, except at the equator and pole, on account of the spheroidal form of the earth; but as the compression of the earth is very small, these directions may be esteemed the same in the present case without sensible error; therefore $r+s-y$ may be regarded as the value of the radius at y. Consequently

$$-d\,y\,.\,rn^2\sin^2 q$$

is the variation of the centrifugal force corresponding to the increased height of the particle; and when compared with $-gd\,y$ the gravity of this little column, it is of the order $\frac{n^2 r}{g}$, the same with the ratio of the centrifugal force to gravity at the equator, or to $\frac{1}{288}$, and therefore may be omitted; hence equation (71) becomes

$$dV - dV' + gd\,y + \frac{n^2}{2}d\,\{(r+s)\sin(q+u)\}^2 = 0\,.$$

277. As the surface of the sea differs very little from that of a sphere, $d\,r$ may be omitted; consequently if

$$\frac{n^2}{2}d\,\{(r+s)\sin(q+u)\}^2$$

be eliminated from equation (70), the result will be

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right) - 2n\sin q\cos q\left(\frac{dv}{dt}\right)\right\}$$

$$+r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\cos q\left(\frac{du}{dt}\right) + 2n\sin^2 q\left(\frac{ds}{dt}\right)\right\} = -gd\,y + dV', \qquad \textbf{(72)}$$

which is the equation of the motion of a particle at the surface of the sea. The variations $d\,y$, dV' correspond to the two variables q and v .

278. To complete the theory of the motions of the atmosphere and ocean, the equation of the continuity of the fluid must now be found.

Continuity of Fluids

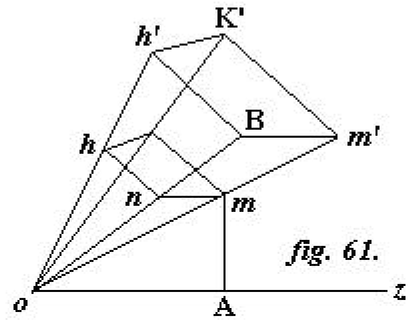


fig. 61.

Suppose $m'h$, fig. 61, to be an indefinitely small rectangular portion of the fluid mass, situate at B, fig. 60, and suppose the solid to be formed by the imaginary rotation of the area $\mathrm{B}nhh'$ about the axis oz; the centre of gravity of $\mathrm{B}nhh'$ will describe an arc, which on account of the smallness of the solid, may without sensible error be represented by mn, its radius being $m\mathrm{A}$; hence the arc mn is $m\mathrm{A}\times mn$. Now the area $\mathrm{B}nhh'$ multiplied by $m\mathrm{A}\times mn$, is equal to the solid $m'h$, supposing it indefinitely small and rectangular.

The colatitude of the point B or $\mathrm{A}om = q$, the longitude of B is $nt + v$, then the indefinitely small increments of these angles are $m'o\mathrm{K}' = dq$, $m'o\mathrm{B} = dv$, for as the figure is independent of the time, nt is constant. Hence if the radii $o\mathrm{B}$, on, be represented by r' and r, the sectors $\mathrm{B}oh'$, noh, are r'^2dq and r^2dq; hence the area

$$\mathrm{B}nhh' = \frac{\left(r'^2 - r^2\right)}{2}dq = \frac{\left(r'+r\right)\left(r'-r\right)}{2}dq\ .$$

But as the thickness is indefinitely small,

$$r'+r = 2r,\ r'-r = dr\,;$$

therefore the area

$$\mathrm{B}nhh' = rdr\,.\,dq\ .$$

Again,

$$\mathrm{A}m = r\sin q\ ,$$

consequently,

$$\mathrm{A}m\,.\,mn = rdv\ \sin q\ ,$$

and thus the volume

$$m'h = r^2drdqdv\ \sin q\ ;$$

and if r be the density,

$$dm = r\,r^2drdq\,dv\ \sin q\ .$$

But in consequence of the disturbing forces, r, q, and v, become $r+s$, $q+u$, $v+v$, after the time t, and dr, dq, dv, become[11]

$$dr+\frac{ds}{dr}\cdot dr,\ dq+\frac{du}{dq}\cdot dq,\ dv+\frac{dv}{dv}\cdot dv\ ;$$

also the density is changed to $r+r'$. If these values be put in the preceding expression for the solid *dm*, it becomes after the time *t* equal to

$$(r+r')(r+s^2)\left(1+\frac{ds}{dr}\right)\left(1+\frac{du}{dq}\right)\left(1+\frac{dv}{dv}\right)dr\,dq\,dv\ \sin(q+u),$$

but this must be equal to the original mass; hence

$$(r+r')(r+s^2)\left(1+\frac{ds}{dr}\right)\left(1+\frac{du}{dq}\right)\left(1+\frac{dv}{dv}\right)\sin(q+u)=rr^2\sin q\ .$$

If the squares and products of

$$s,\ \frac{ds}{dr},\ \frac{du}{dq},\ \frac{dv}{dv}$$

be omitted, and observing that

$$2rs+r^2\frac{ds}{dr}=\frac{d\cdot r^2 s}{dr},$$

and

$$\sin(q+u)=\sin q+u\cos q\ ;$$

for as *u* is very small, the arc may be put for the sine, and unity for the cosine, the equation of the continuity of the fluid is[12]

$$0=r^2(r+r')\left\{\left(\frac{du}{dq}\right)+\left(\frac{dv}{dv}\right)+\frac{u\cos q}{\sin q}\right\}+r\left(\frac{d\cdot r^2 s}{dr}\right), \qquad \textbf{(73)}$$

expressed in polar co-ordinates.

279. The equations (70), (72), and (73), are perfectly general; and therefore will answer either for the oscillations of the ocean or atmosphere.

Oscillations of the Ocean

280. The density of the sea is constant, therefore $r'=0;$ hence the equation of continuity becomes

$$0 = \left(\frac{d\,.r^2 s}{dr}\right) + r^2 \left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cos q}{\sin q}\right\}.$$

In order to find the integral of this equation with regard to r only, it may be assumed, that all the particles that are on any one radius at the origin of the time, will remain on the same radius during the motion; therefore r, v, and u will be nearly the same on the small part of the terrestrial radius between the bottom and surface of the sea; hence, the integral will be

$$0 = r^2 s - \left(r^2 s\right) + r^2 g \left\{\left(\frac{du}{dq}\right) + \frac{dv}{dv} + \frac{u\cos q}{\sin q}\right\}$$

$(r^2 s)$ is the value of $r^2 s$ at the surface of the sea, but the change in the radius of the earth between the bottom and surface of the sea is so small, that $r^2(s)$ may be put for $(r^2 s)$; then dividing the whole by r^2, and neglecting the terms $\frac{2g(s)}{r}$, which is the ratio of the depth of the sea to the terrestrial radius, and therefore very small, the mean depth even of the Pacific ocean being only about four miles, whereas the mean radius of the earth is nearly 4,000 miles; the preceding equation becomes

$$0 = s - (s) + g \left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cos q}{\sin q}\right\}. \qquad \textbf{(74)}$$

Now $g + s - (s)$ is the whole depth of the sea from the bottom to the highest point to which the tides rise at its surface of momentary equilibrium; and g varies with the angles v and q; hence at the surface of equilibrium, it becomes

$$g + u\frac{du}{dq} + v\frac{dg}{dv};$$

and as y is the height of a particle above the surface of equilibrium, it follows that

$$g + s - (s) = -y + g + u\frac{dg}{dq} + v\frac{dg}{dv},$$

or

$$s - (s) = -y + u\frac{dg}{dq} + v\frac{dg}{dv}.$$

Whence the equation of continuity becomes

$$y = -\frac{d\left(gu\right)}{dq} - \frac{d\left(gv\right)}{dv} - \frac{gu\cos q}{\sin q}. \qquad \textbf{(75)}$$

281. In order to apply the other equations to the motion of the sea, it must be observed that a fluid particle at the bottom of the sea would in its rotation from m to B always touch the spheroid, which is nearly a sphere; therefore the value of s would be very small for that particle, and would be to v, u, of the order of the eccentricity of the spheroid, to its mean radius taken as unity; consequently at the bottom of the sea, s may be omitted in comparison of u, v. But it appears from equations (74), that $s-(s)$ is a function of u and v, independent of r, when the very small quantity $\frac{2g\,(s)}{r}$ is omitted: hence s is the same throughout every part of the radius r, as it is at the bottom, and may therefore be omitted throughout the whole depth, when compared with u and v, so that equation (72) of the surface of the fluid is reduced to[13]

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right)-2n\sin q\,\cos q\left(\frac{dv}{dt}\right)\right\} \qquad \textbf{(76)}$$
$$+r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right)+2n\sin q\,\cos q\left(\frac{du}{dt}\right)\right\}=-gdy+dV',$$

282. When the fluid mass is in momentary equilibrium, the equation for the motion of a particle in the interior of the fluid becomes

$$0=\tfrac{1}{2}n^2 d\left\{(r+s)\sin(q+u)\right\}^2+(dV)-\frac{(dp)}{r},$$

where (dV), (dp), are the values of dV and dp suited to that state. But we may suppose that in a state of motion,

$$dV=(dV)+dV',\ \text{and}\ dp=(dp)+dp';$$

whence

$$(dV)=dV-dV',\ (dp)=dp-dp',$$

and

$$\tfrac{1}{2}n^2 d\left\{(r+s)\sin(q+u)\right\}^2=dV'-dV+\frac{dp}{r}-\frac{dp'}{r}.$$

283. If the first member of this expression be eliminated from equation (70), with regard to the independent variation of r alone, it gives

$$\frac{d\left(V'-\frac{p'}{r}\right)}{dr}=\left(\frac{d^2s}{dt^2}\right)-2nr\sin^2 q\left(\frac{dv}{dt}\right). \qquad \textbf{(77)}$$

284. Now $n\left(\frac{dv}{dt}\right)$ is of the order y, s, or $\frac{gs}{r}$; for if the coefficients of dq, dv, be each made zero in equation (76), it will give

$$r^2\left(\frac{d^2u}{dt^2}\right)-2nr^2\sin q\cos q\left(\frac{dv}{dt}\right)=-g\left(\frac{dy}{dq}\right)+\left(\frac{dV'}{dq}\right),$$
$$r^2\sin^2 q\left(\frac{d^2v}{dt^2}\right)+2nr^2\sin q\cos q\left(\frac{du}{dt}\right)=-g\left(\frac{dy}{dv}\right)+\left(\frac{dV'}{dv}\right);$$

add the differential of the last equation relative to t, to the first equation multiplied by

$$-2n\sin q\cos q$$

and let the second member of this equation be represented by

$$y'.r^2\sin^2 q\ ,$$

then divide by

$$r^2\sin^2 q\ ,$$

and put

$$2n\cos q=a,$$

and there will be found the linear equation

$$\left(\frac{d^2v}{dt^2}\right)+a^2\left(\frac{dv}{dt}\right)=y'.$$

The value of $\frac{dv}{dt}$ obtained from the integral of this equation will be a function of y', and as y' is a function of y and V', each of which is of the order s or $\frac{gs}{r}$, $\frac{dv}{dt}$; consequently

$$\frac{d\left(V'-\frac{p'}{r}\right)}{dr}$$

is of the same order. If then equation (77), be multiplied by dr its integral will be

$$V'-\frac{p'}{r}=\int dr\left\{\left(\frac{d^2s}{dt^2}\right)-2nr\sin^2 q\left(\frac{dv}{dt}\right)\right\}+l\ .$$

285. Since this equation has been integrated with regard to r only, l must be a function of q, v, and t, independent of r, according to the theory of partial equations. And as the function in r is of the order $\frac{g s}{r}$ it may be omitted; and then

$$dV' - \frac{d p'}{r} = d l \,,$$

by which equation (70) becomes[14]

$$r^2 dq \left\{ \left(\frac{d^2 u}{dt^2} \right) - 2n \sin q \cos q \left(\frac{dV}{dt} \right) \right\}$$
$$+ r^2 dv \left\{ \sin^2 q \left(\frac{d^2 v}{dt^2} \right) + 2n \sin q \cos q \left(\frac{du}{dt} \right) \right\} = d l \,.$$

286. But as $d l$ does not contain r, s, or y, it is independent of the depth of the particle; hence this equation is the same for a particle at the surface, or in its neighbourhood, consequently it must coincide with equation (76); and therefore

$$d l = dV' - g d y \,.$$

287. Thus it appears, that the whole theory of the tides would be determined if integrals of the equations

$$r^2 dq \left\{ \left(\frac{d^2 u}{dt^2} \right) - 2n \sin q \cos q \left(\frac{dv}{dt} \right) \right\}$$
$$+ r^2 dv \left\{ \sin^2 q \left(\frac{d^2 v}{dt^2} \right) + 2n \sin q \cos q \left(\frac{du}{dt} \right) \right\} = -g d y + dV'$$
$$y = -\frac{d(g u)}{dq} - \frac{d(g v)}{dv} - \frac{g u \cos q}{\sin q}$$

could be found, for the horizontal flow might be obtained from the first, by making the coefficients of the independent quantities dq, dv, separately zero, then the height to which they rise would be found from the second. This has not yet been done, as none of the known methods of analysis have hitherto succeeded.

288. These equations have been formed on the hypothesis of the earth being entirely covered by the sea; hence the integrals, if they could be found, would be inadequate to determine the oscillations of the ocean retarded or accelerated by the continents, islands, and innumerable other causes, beyond the reach of analysis. No attempt is therefore made to integrate the equations; but the theory of the tides is determined by comparing the general relations which subsist between the observed phenomena and the causes which produce them.

289. In order to integrate the equation of continuity, it was assumed that if the angles P*ob*, *m*P*b*, [fig. 60] or rather

$$u,\ \frac{du}{dt},\ v,\ \frac{dv}{dt},$$

be the same for every particle situate on the same radius throughout the whole depth of the sea at the beginning of the motion, they will always continue to be the same for that set of particles during their motion, therefore all the fluid particles that are at the same instant on any one radius, will continue very nearly on that radius during the oscillations of the fluid. Were this rigorously true, the horizontal flow of the tides would be isochronous, like the oscillations of a pendulum, and their velocity would be inversely as their depth, provided the particles had no motion in latitude; and it may be nearly so in the Pacific, whose mean depth is about four miles, and where the tides only rise to about five feet; but it is very far from being the case in shallow seas, and on the coasts where the tides are high; because the condition of isochronism depends on the omission of quantities of the order of the ratio of the height of the tides to the depth of the sea.

290. The reaction of the sea on the terrestrial spheroid is so small that it is omitted. The common centre of gravity of the spheroid and sea is not changed by this reaction, and therefore the ratio of the action of the sea on the spheroid, is to the reaction of the spheroid on the sea, as the mass of the sea to the solid mass; that is, as the depth of the sea to the radius of the earth, or at most as 1 to 1000, assuming the mean depth of the sea to be four miles. For that reason u, v, express the true velocity of the tides in longitude and latitude, as they were assumed to be.

On the Atmosphere

291. Experience shows the atmosphere to be an elastic fluid, whose density increases in proportion to the pressure. It is subject to changes of density from the variation of temperature in different latitudes, at different heights, and from various other causes; but in this investigation the temperature is assumed to be constant.

292. Since the air resists compression equally in all directions, the height of the atmosphere must be unlimited if its atoms be infinitely divisible. Some considerations, however, induced Dr. Wollaston[15] to suppose that the earth's atmosphere is of finite extent, limited by the weight of ultimate atoms of definite magnitude, no longer divisible by repulsion of their parts. But whether the particles of the atmosphere be infinitely divisible or not, all phenomena concur in proving its density to be quite insensible at the height of about fifty miles.

Density of the Atmosphere

293. The law by which the density of the air diminishes as the height above the surface of the sea increases, will appear by considering ρ, ρ', ρ'', to be the densities of three contiguous strata of air, the thickness of each being so small that the density may be assumed uniform throughout each stratum. Let p be the pressure of the superincumbent air on the lowest stratum, p' the pressure on the next, and p'' the pressure on the third; and let m be a coefficient, such that $r = a\,p$. Then, because the densities are as the pressures,

$$r' = a\,p', \text{ and } r'' = a\,p''.$$

Hence,[16]

$$r - r' = a\,(p - p') \text{ and } r' - r'' = a\,(p' - p'').$$

But $p - p'$ is equal to the weight of the first of these strata, and $p' - p''$ is equal to that of the second: hence

$$r - r' : r' - r'' :: r : r';$$

consequently

$$r\,r'' = r'^2.$$

The density of the middle stratum is therefore a mean proportional between the densities of the other two; and whatever be the number of equidistant strata, their densities are in continual proportion.

294. If the heights therefore, from the surface of the sea, be taken in an increasing arithmetical progression, the densities of the strata of air will increase in geometrical progression, a property that logarithms possess relatively to their numbers.

295. All the circumstances both of the equilibrium and motion of the atmosphere may be determined from equation (70), if the quantities it contains be supposed relative to that compressible fluid instead of to the ocean.

Equilibrium of the Atmosphere

296. When the atmosphere is in equilibrio v, u, and s are zero, which reduces equation (70) to

$$\frac{n^2}{2}\cdot r^2 \,.\, \sin^2 q + V - \int \frac{d\,p}{r} = \text{constant}.$$

Suppose the atmosphere to be every where of the same density as at the surface of the sea, let h be the height of that atmosphere which is very small, not exceeding $5\frac{1}{2}$ miles, and let g be the force of gravity at the equator; then as the pressure is proportional to the density, $p = h\,.\,g\,.\,r$, and[17]

$$\int \frac{dp}{\rho} = hg\,.\log \rho\ ,$$

consequently the preceding equation becomes

$$hg\,.\log \rho = \text{constant} + V + \frac{n^2}{2}\cdot r^2\,.\sin^2\theta\ .$$

At the surface of the sea, V is the same for a particle of air, and for the particle of the ocean adjacent to it; but when the sea is in equilibrio

$$V + \frac{n^2}{2}\cdot r^2\,.\sin^2\theta = \text{constant}\ ,$$

therefore ρ is constant, and consequently the stratum of air contiguous to the sea is every where of the same density.

297. Since the earth is very nearly spherical, it may be assumed that r the distance of a particle of air from its centre is equal to $R + r'$, R being the terrestrial radius extending to the surface of the sea, and r' the height of the particle above that surface. V, which relates to the surface of the sea, becomes at the height r';

$$V' = V + r'\left(\frac{dV}{dr}\right) + \&c.$$

by Taylor's theorem,[18] consequently the substitution of $R + r'$ for r in the value of[19] $hg\,.\log \rho$ gives

$$hg\cdot\log \rho = \text{constant} + V + r'\left(\frac{dV}{dr}\right) + \frac{r'^2}{2}\left(\frac{d^2V}{dr^2}\right) + \frac{n^2}{2}\cdot R^2\cdot\sin^2\theta + n^2\cdot Rr'\cdot\sin^2\theta$$

V, $\left(\frac{dV}{dr}\right)$, &c. relate to the surface of the sea where

$$V + \frac{n^2}{2}\cdot R^2\,.\sin^2\theta = \text{constant}\,,$$

and as

$$-\left(\frac{dV}{dr}\right) - n^2\cdot R\cdot\sin^2\theta\ ,$$

is the effect of gravitation at that surface, it may be represented by g', whence

$$hg \cdot \log r = \text{constant} - r'g' + \frac{r'^2}{2} \cdot \left(\frac{d^2V}{dr^2}\right).$$

298. Since $\left(\frac{d^2V}{dr^2}\right)$ is multiplied by the very small quantity r'^2, it may be integrated in the hypothesis of the earth being a sphere; but in that case

$$-\left(\frac{dV}{dr}\right) = g' - \frac{m}{R^2}$$

m being the mass of the earth; hence

$$\left(\frac{d^2V}{dr^2}\right) = -\frac{2m}{R^2} = -\frac{2g'}{R};$$

consequently the preceding equation becomes

$$\log r = -\frac{r'}{h} \cdot \frac{g'}{g}\left(1 + \frac{r'}{R}\right);$$

whence

$$r = r'.c^{-\frac{r'g'}{hg} \cdot \left(1 + \frac{r'}{R}\right)};$$

an equation which determines the density of the atmosphere at any given height above the level of the sea; c is the number whose logarithm is unity, and r' a constant quantity equal to the density of the atmosphere at the surface of the sea.

299. If g' and *g* be the force of gravity at the equator and in any other latitude, they will be proportional to l' and *l*, the lengths of the pendulum beating seconds at the level of the sea in these two places; hence l' and *l*, which are known by experiment, may be substituted for g' and *g*, and the formula becomes[20]

$$r = r'.c^{-\frac{r'l'}{hl} \cdot \left(1 + \frac{r'}{R}\right)}. \qquad \textbf{(78)}$$

Whence it appears that strata of the same density are every where very nearly equally elevated above the surface of the sea.

300. By this expression the density of the air at any height may be found, say at fifty-five miles. $\frac{r'}{R}$ is very small and may be neglected; and *l* may be made equal to l' without sensible error; hence

$$r = r'c^{-\frac{r'}{h}}.$$

Now the height of an atmosphere of uniform density is only about $h = 5\frac{1}{2}$ English miles; hence if

$$r' = 10h = 55, \; r = r'c^{-10},$$

and as[21]

$$c = 2.71828, \; r = \frac{r'}{22{,}026},$$

so that the density at the height of 55 English miles is extremely small, which corresponds with what was said in article 292.

301. From the same formula the height of any place above the level of the sea may be found; for the densities r' and r, and consequently *h*, are given by the height of the barometer, l' and l, the lengths of the seconds' pendulum for any latitude are known by experiment; and *R*, the radius of the earth is also a given quantity; hence r' may be found. But in estimating the heights of mountains by the barometer, the variation of gravity at the height r' above the level of the sea cannot be omitted, therefore $\frac{l'-l}{l'}r'$ must be included in the preceding formula.

Oscillations of the Atmosphere

302. The atmosphere has the form of an ellipsoid flattened at the poles, in consequence of its rotation with the earth, and its strata by article 299, are everywhere of the same density at the same elevation above the surface of the sea. The attraction of the sun and moon occasions tides in the atmosphere perfectly similar to those of the ocean; however, they are probably affected by the rise and the fall of the sea.

303. The motion of the atmosphere is determined by equations (70), (73), which give the tides of the ocean, with the exception of a small change owing to the elasticity of the air; hence the term $\frac{dp}{r}$, expressing the ratio of the pressure to the density cannot be omitted as it was in the case of the sea.

Let $r = (r) + r'$; (r) being the density of the stratum in equilibrio, and r' the change suited to a state of motion; hence

$$p = hg\big((r) + r'\big),$$

and

$$\frac{dp}{r} = hg\frac{d(r)}{(r)} + g\frac{d(hr')}{(r)}.$$

Let

$$\frac{hr'}{(r)} = y',$$

then

$$\frac{dp}{r} = hg\frac{d(r)}{(r)} + gdy'.$$

304. The part $hg\frac{d(r)}{(r)}$ vanishes, because in equilibrio

$$\frac{n^2}{2}d\left\{(r+s)\sin(q+u)\right\}^2 + (dV) - hg\frac{d(r)}{(r)} = 0,$$

therefore

$$\frac{dp}{r} = gdy'.$$

Let f be the elevation of a particle of air above the surface of equilibrium of the atmosphere which corresponds with y, the elevation of a particle of water above the surface of equilibrium of the sea. Now at the sea $f = y$, for the adjacent particles of air and water are subject to the same forces; but it is necessary to examine whether the supposition of $f = y$, and of y being constant for all the particles of air situate on the same radius are consistent with the equation of continuity (73), which for the atmosphere is

$$0 = r^2\left\{r' + (r)\left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cdot\cos q}{\sin q}\right\}\right\} + (r)\cdot\left(\frac{d\cdot r^2 s}{dr}\right).$$

If the value of $\frac{r'}{(r)}$ from this equation be substituted in $\frac{hr'}{(r)} = y'$, it becomes[22]

$$y' = -h\cdot\left\{\left(\frac{d.r^2 s}{r^2 . dr}\right) + \left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u.\cos q}{\sin q}\right\}.$$

The part of s that depends on the variation of the angles q and v is so small, that it may be neglected, consequently $s = f$; and if $f = y$ then $\left(\frac{df}{dr}\right) = 0$, since[23] the value of f is the same for all the particles situate on the same radius. Also y is of the order h or $\frac{n^2}{g}$; consequently[24]

$$y' = -h\cdot\left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cdot\cos q}{\sin q}\right\}, \qquad \textbf{(79)}$$

then u and v being the same for all the particles situate primitively on the same radius, the value of y' will be the same for all these particles, and as quantities of the order ds are omitted, equation (70) becomes

$$r^2 dq \left\{ \left(\frac{d^2u}{dt^2} \right) - 2n \sin q \cos q \left(\frac{dv}{dt} \right) \right\}$$
$$+ r^2 dv \left\{ \sin^2 q \left(\frac{d^2v}{dt^2} \right) + 2n \sin q \cos q \left(\frac{du}{dt} \right) \right\} \qquad \textbf{(80)}$$
$$= dV - g\,dy' - g\,dy\,.$$

Thus the equations that determine the oscillations of the atmosphere only differ from those that give the tides by the small quantity $g\,dy'$, depending on the elasticity of the air.

305. Finite values of the equations of the motion of the atmosphere cannot be obtained; therefore the ebb and flow of the atmosphere may be determined in the same manner as the tides of the ocean, by estimating the effects of the sun and moon separately. This can only be effected by a comparison of numerous observations.

Oscillations of the Mercury in the Barometer

306. Oscillations in the atmosphere cause analogous oscillations in the barometer. For suppose a barometer to be fixed at any height above the surface of the sea, the height of the mercury is proportional to the pressure on that part of its surface that is exposed to the action of the air. As the atmosphere rises and falls by the action of the disturbing forces like the waves of the sea, the surface of the mercury is alternately more or less pressed by the variable mass of the atmosphere above it. Hence the density of the air at the surface of the mercury varies for two reasons; first, because it belonged to a stratum which was less elevated in a state of equilibrium by the quantity *y*, and secondly, because the density of a stratum is augmented when in motion by the quantity $\frac{(r)}{h} \cdot y$. Now if *h* be the height of the atmosphere in equilibrio when its density is uniform, and equal to (r), then

$$h : y :: (r) : y \cdot \frac{(r)}{h},$$

the increase of density in a state of motion from the first cause. In the same manner, $y' \cdot \frac{(r)}{h}$ is the increase of density from the second cause. Thus the whole increase is

$$(r) \frac{(y' + y)}{h}.$$

And if *H* be the height of the mercury in the barometer when the atmosphere is in equilibrio, its oscillations when in motion will be expressed by

$$H\frac{(y'+y)}{h}. \qquad \textbf{(81)}$$

The oscillations of the mercury are therefore similar at all heights above the level of the sea, and proportional in their extent to the height of' the barometer.

Conclusion

307. The account of the first book of the *Mécanique Céleste*[25] is thus brought to a conclusion. Arduous as the study of it may seem, the approach in every science, necessarily consisting in elementary principles, must be tedious: but let it not be forgotten, that many important truths, coeval[26] with the existence of matter itself, have already been developed; and that the subsequent application of the principles which have been established, will lead to the contemplation of the most sublime works of the Creator. The general equation of motion has been formed according to the primordial laws of matter; and the universal application of this one equation, to the motion of matter in every form of which it is susceptible, whether solid or fluid, to a single particle, or to a system of bodies, displays the essential nature of analysis, which comprehends every case that can result from a given law. It is not, indeed, surprising that our limited faculties do not enable us to derive general values of the unknown quantities from this equation: it has been accomplished, it is true, in a few cases, but we must be satisfied with approximate values in by much the greater number of instances. Several circumstances in the solar system materially facilitate the approximations; these Laplace[27] has selected with profound judgment, and employed with the greatest dexterity.

Notes

[1] Two equations numbered 56 occur in the 1st edition. We retain this numbering.
[2] Poisson, Siméon Denis, 1781-1840, mathematician, born in Pithiviers, France. From 1806 Poisson was a professor at the École Polytechnique. His contributions included work in electromagnetism and probability and a law governing the distribution of randomly occurring events (the Poisson distribution). He is known best for his work in celestial mechanics through his *Traité de mécanique* (1811 & 1833). Poisson extended the work of Lagrange (see note 16, *Preliminary Dissertation*) and Laplace (see note 4, *Foreword to the Second Edition*) on the stability of planetary orbits and calculated the attraction of both spheroids and ellipsoids. He was also known for his work on the expression of planetary gravitational force as a function of mass distribution within the planet.
[3] Punctuation changed from a semicolon in the 1st edition.
[4] Period added.
[5] Period added.
[6] Period added.
[7] Middle left hand term in 1st edition contains an error written $\frac{ds}{dt}dy$.
[8] See note 35, *Preliminary Dissertation.*
[9] *Aries.* In astronomy, an autumn zodiacal constellation located between Pisces and Taurus. The first point of Aries, or vernal equinox, is an intersection of the celestial equator with the apparent annual pathway of the Sun and the point in the sky from which celestial longitude and right ascension are measured.

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10 The first term contains an unmatched bracket and *reads* $r^2 dq\left\{\left(\frac{d^2u}{dt^2} - 2n\sin q\ \cos\left(\frac{dv}{dt}\right)\right\}\right.$ in the 1st edition.

11 The first expression below *reads* $dr + \frac{ds}{dr}dr$ in the 1st edition.

12 Equation (73) *reads* with an unmatched bracket as $0 = r^2\,(r + r'\left(\left(\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cos q}{\sin q}\right) + r\left(\frac{d\cdot r^2 s}{dr}\right)\right.$ in the 1st edition. The form used here is in conformity with the presentation in article 280.

13 A comma after the first term below in the 1st edition is not used here.

14 Comma after first term in original text is not used here.

15 See note 62, *Preliminary Dissertation.*

16 Error in 1st edition *reads* $r - r = a\ (p' - p'')$ for the second expression.

17 Equation below *reads* $\int \frac{dp}{r} = hg \cdot \log\cdot\ r$ in the 1st edition.

18 *Taylor series.* If f possesses derivatives of all orders at $x = c$ and is represented by $\sum_{n=0}^{\infty} a_n\,(x-c)^n$ in an interval of convergence of positive radius, then we must have $a_0 = f\,(c)$, and $a_n = \frac{f^{(n)}(c)}{n!}(n = 1,2,3,\ldots)$, so that

$$f\,(x) = f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

This is called the *formal Taylor series* for f in powers of $(x-c)$. Fobes and Smyth, *Calculus and Analytic Geometry*, V.2, Prentice Hall, 1963.

19 This *reads* $hg\ \log.r$ in the 1st edition. The form we use is consistent with the use in the next equation.

20 Punctuation added at end of equation.

21 Comma separator added in denominator. The 1st edition reads $r = \frac{r'}{22026}$.

22 Punctuation added at end of next equation.

23 Punctuation changed here. The 1st edition reads: "…then $\left(\frac{df}{dr}\right) = 0$. Since …"

24 Comma added at end of next equation.

25 See note 4, *Introduction.*

26 Coinciding in time of origin or existence; contemporary. *The Wordsmyth Educational Dictionary-Thesaurus.*

27 See note 25 above.