

BOOK I

CHAPTER VI

ON THE EQUILIBRIUM OF FLUIDS

Definitions, &c.

225. A FLUID is a mass of particles, which yield to the slightest pressure, and transmit that pressure in every direction.

226. Mobility of the particles constitutes the difference between fluids and solids.

227. There are, indeed, fluids in nature whose particles adhere more or less to each other, called viscous fluids; but those only whose particles do not adhere in any degree, but possess perfect mobility, are the subject of this investigation.

228. Strictly speaking, all fluids are compressible, for even liquids under very great pressure change their volume; but as the compression is insensible in ordinary circumstances, fluids of perfect mobility are divided into compressible or elastic fluids, and incompressible.

229. The elastic and compressible fluids are atmospheric air, the gases, and steam. When compressed, these fluids change both form and volume, and regain their primitive state as soon as the pressure is removed. Some of the gases are found to differ from atmospheric air in losing their elastic form, and becoming liquid when compressed to a certain degree, as lately proved by Mr. Faraday,^{1 2} and steam is reduced to water when its temperature is diminished; but atmospheric air, and others of the gases, always retain their gaseous form, whatever the degree of pressure may be.

230. It is impossible to ascertain the forms of the particles of fluids, but as all of them, considered in mass, afford the same phenomena, it can have no influence on the laws of their motions.

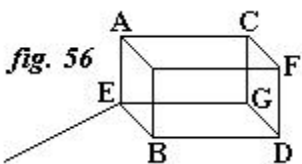
Equilibrium of Fluids

231. When a fluid is in equilibrio, each particle must itself be held in equilibrio by the forces acting upon it, together with the pressures of the surrounding particles.

232. It is evident, that whatever the accelerating forces or pressures may be, they can all be resolved into component forces parallel to three rectangular co-ordinates, ox , oy , oz .

Equation of Equilibrium

233. Imagine a system of fluid particles, forming a rectangular parallelepipedon³ A B C D, fig. 56, and suppose its sides parallel to the co-ordinate axes. Suppose also, that it is pressed on all sides by the surrounding fluid, at the same time that it is urged by accelerating forces.



234. It is evident, that the pressure on the face A B, must be in a contrary direction to the pressure on the face C D; hence the mass will be urged by the difference of these pressures: but this difference may be considered as a single force acting either on the face A B, or C D; consequently the difference of the pressures multiplied by the very small area A B will be the whole pressure, urging the mass parallel to the side E G. In the same manner, the pressures urging the mass in a direction parallel to E B and E A, are the area E C into the difference of the pressures on the faces E C and B F; and the area E D into the difference of the pressures on E D and A F.

235. Because the mass is indefinitely small, if x, y, z , be the coordinates of E, the edges E G, E B, E A, may be represented by dx, dy, dz . Then p being the pressure on a unit of surface, $pdydz$ will be the pressure on the face A B, in the direction E G. At G, x becomes $x + dx$, y and z remaining the same; hence as p is considered a function of x, y, z , it becomes

$$p' = p + \left(\frac{dp}{dx} \right) dx \text{ at the point G ;}$$

hence

$$p - p' = - \left(\frac{dp}{dx} \right) dx ,$$

and

$$pdydz - p'dydz = - \left(\frac{dp}{dx} \right) dx \cdot dydz .$$

Now $pdydz$ is the pressure on A B, and $p'dydz$ is the pressure on C D; hence

$$- \left(\frac{dp}{dx} \right) dx \cdot dydz = (p - p') dydz$$

is the difference of the pressures on the faces A B and C D. In the same manner it may be proved that

$$- \left(\frac{dp}{dy} \right) dy \cdot dx dz, \text{ and } - \left(\frac{dp}{dz} \right) dz \cdot dy dx$$

are the differences of the pressures on the faces B F, A G, and on E D, A F.

236. But if X, Y, Z , be the accelerating forces in the direction of the axes, when multiplied by the volume $dx dy dz$, and by \mathbf{r} its density, they become the momenta

$$\begin{aligned} &\mathbf{r} \cdot X dx dy dz, \\ &\mathbf{r} \cdot Y dx dy dz, \\ &\mathbf{r} \cdot Z dx dy dz. \end{aligned}$$

But these momenta must balance the pressures in the same directions when the fluid mass is in equilibrio; hence, by the principle of virtual velocities

$$\left\{ \mathbf{r}X - \frac{dp}{dx} \right\} \mathbf{d}x + \left\{ \mathbf{r}Y - \frac{dp}{dy} \right\} \mathbf{d}y + \left\{ \mathbf{r}Z - \frac{dp}{dz} \right\} \mathbf{d}z = 0,$$

or

$$\frac{dp}{dx} \mathbf{d}x + \frac{dp}{dy} \mathbf{d}y + \frac{dp}{dz} \mathbf{d}z = \mathbf{r} \{ X \mathbf{d}x + Y \mathbf{d}y + Z \mathbf{d}z \}.$$

As the variations are arbitrary, they may be made equal to the differentials, and then

$$dp = \mathbf{r} \{ X dx + Y dy + Z dz \} \tag{52}$$

is the general equation of the equilibrium of fluids, whether elastic or incompressible. It shows, that the indefinitely small increment of the pressure is equal to the density of the fluid mass multiplied by the sum of the products of each force by the element of its direction.

237. This equation will not give the equilibrium of a fluid under all circumstances, for it is evident that in many cases equilibrium is impossible; but when the accelerating forces are attractive forces directed to fixed centres, it furnishes another equation, which shows the relation that must exist among the component forces, in order that equilibrium may be possible at all. It is called an equation of condition, because it expresses the general condition requisite for the existence of equilibrium.

Equations of Condition

238. Assuming the forces X, Y, Z , to be functions of the distance, by article 75⁴ the second member or the preceding equation is an exact differential; and as p is a function of x, y, z , it gives the partial equations

$$\frac{dp}{dx} = \mathbf{r}X; \quad \frac{dp}{dy} = \mathbf{r}Y; \quad \frac{dp}{dz} = \mathbf{r}Z;$$

but the differential of the first, according to y , is

$$\frac{d^2 p}{dx dy} = \frac{d \cdot rX}{dy}$$

and the differential of the second, according to x , is

$$\frac{d^2 p}{dy dx} = \frac{d \cdot rY}{dx};$$

hence

$$\frac{d \cdot rX}{dy} = \frac{d \cdot rY}{dx}.$$

By a similar process, it will be found that

$$\frac{d \cdot rY}{dz} = \frac{d \cdot rZ}{dy}; \quad \frac{d \cdot rX}{dz} = \frac{d \cdot rZ}{dx}.$$

These three equations of condition are necessary, in order that the equation (52) may be an exact differential, and consequently integrable. If the differentials of these three equations be taken, the sum of the first multiplied by Z , of the second multiplied by X , and of the third multiplied by $-Y$, will be

$$0 = X \cdot \frac{dY}{dz} - Y \cdot \frac{dX}{dz} + Z \cdot \frac{dX}{dy} - X \cdot \frac{dZ}{dy} + Y \cdot \frac{dZ}{dx} - Z \cdot \frac{dY}{dx}$$

an equation expressing the relation that must exist among the forces X , Y , Z , in order that equilibrium may be possible.

Equilibrium will always be possible when these conditions are fulfilled; but the exterior figure of the mass must also be determined.

Equilibrium of homogeneous Fluids

239. If the fluid be free at its surface, the pressure must be zero in every point of the surface when the mass is in equilibrium; so that $p = 0$, and

$$r\{Xdx + Ydy + Zdz\} = 0,$$

whence

$$\int (Xdx + Ydy + Zdz) = \text{constant},$$

supposing it an exact differential, the density being constant.

The resulting force on each particle must be directed to the interior of the fluid mass, and must be perpendicular to the surface; for were it not, it might be resolved into two others, one

perpendicular, and one horizontal; and in consequence of the latter, the particle would slide along the surface.

If $u=0$ be the equation of the surface, by article 69 the equation of equilibrium at the surface will be

$$Xdx + Ydy + Zdz = I du ,$$

I being a function x, y, z ; and by the same article, the resultant of the forces X, Y, Z , must be perpendicular to those parts of the surface where the fluid is free, and the first member must be an exact differential.

Equilibrium of heterogeneous Fluids

240. When the fluid mass is heterogeneous, and when the forces are attractive, and their intensities functions of the distances of the points of application from their origin, then the density depends on the pressure; and all the strata or layers of a fluid mass in which the pressure is the same, have the same density throughout their whole extent.

Demonstration. Let the function

$$Xdx + Ydy + Zdz$$

be an exact difference, which by article 75 will always be the case when the forces X, Y, Z , are attractive, and their intensities functions of the mutual distances of the particles. Assume

$$f = \int (Xdx + Ydy + Zdz), \tag{53}$$

f being a function of x, y, z ; then equation (52) becomes

$$dp = r . df . \tag{54}$$

The first member of this equation is an exact differential, and in order that the second member may also be an exact differential, the density r must be a function of f . The pressure p will then be a function of f also; and the equation of the free surface of the fluid will be $f = \text{constant}$ quantity, as in the case of homogeneity. Thus the pressure and the density are the same for all the points of the same layer. The law of the variation of the density in passing from one layer to another depends on the function in f which expresses it. And when that function is given, the pressure will be obtained by integrating the equation $dp = r df$.

241. It appears from the preceding investigation, that a homogeneous liquid will remain in equilibrio, if all its particles act on each other, and are attracted towards any number of fixed centres; but in that case, the resulting force must be perpendicular to the surface of the liquid, and must tend to its interior. If there be but one force or attraction directed to a fixed point, the

mass would become a sphere, having that point in its centre, whatever the law of the force might be.

242. When the centre of the attractive force is at an infinite distance, its direction becomes parallel throughout the whole extent of the fluid mass; and the surface, when in equilibrio, is a plane perpendicular to the direction of the force. The surface of a small extent of stagnant water may be estimated plane, but when it is of great extent, its surface exhibits the curvature of the earth.

243. A fluid mass that is not homogeneous but free at its surface will be in equilibrio, if the density be uniform throughout each indefinitely small layer or stratum of the mass, and if the resultant of all the accelerating forces acting on the surface be perpendicular to it, and tending towards the interior. If the upper strata of the fluid be most dense, the equilibrio will be unstable; if the heaviest is undermost, it will be stable.

244. If a fixed solid of any form be covered by fluid as the earth is by the atmosphere, it is requisite for the equilibrio of the fluid that the intensity of the attractive forces should depend on their distances from fixed centres, and that the resulting force of all the forces which act at the exterior surface should be perpendicular to it, and directed towards the interior.

245. If the surface of an elastic fluid be free, the pressure cannot be zero till the density be zero; hence an elastic fluid cannot be in equilibrio unless it be either shut up in a close vessel, or, like the atmosphere, it extend in space till its density becomes insensible.

Equilibrium of Fluids in Rotation

246. Hitherto the fluid mass has been considered to be at rest; but suppose it to have a uniform motion of rotation about a fixed axis, as for example the axis *oz*. Let w be the velocity of rotation common to all the particles of the fluid, and r the distance of a particle dm from the axis of rotation, the co-ordinates of dm being x, y, z . Then wr will be the velocity of dm , and its centrifugal force resulting from rotation, will be w^2r , which must therefore be added to the accelerating forces which urge the particle; hence equation (53) will be

$$d\mathbf{f} = Xdx + Ydy + Zdz + w^2rdr .$$

And the differential equation of the strata, and of the free surface of the fluid, will be

$$Xdx + Ydy + Zdz + w^2 . rdr = 0 . \tag{55}$$

The centrifugal force, therefore, does not prevent the function \mathbf{f} from being an exact differential, consequently equilibrio will be possible, provided the condition of article 238 be fulfilled.

247. The regularity of gravitation at the surface of the earth; the increase of density towards its centre; and, above all, the correspondence of the form of the earth and planets with

that of a fluid mass in rotation, have led to the supposition that these bodies may have been originally fluid, and that their parts, in consolidating, have retained nearly the form they would have acquired from their mutual attractions, together with the centrifugal force induced by rotation when fluid. In this case, the laws expressed by the preceding equations must have regulated their formation.

Notes

¹ Faraday, Michael, 1791-1867, chemist, experimental physicist, and natural philosopher, born in Newington Butts, England. Faraday's research included work on the condensation of gases, the conservation of force, and studies on benzene and steel. His major work is the series of *Experimental Researches on Electricity* (1839-55), in which he reports discoveries about electricity, electrolysis, and relationships between electricity and magnetism. Faraday and Somerville corresponded frequently. In the first edition of her *On the Connexion of the Physical Sciences* (see note 39, *Foreword to the Second Edition*) Mary Somerville was one of the first to report on Faraday's recent researches on the voltaic pile. Faraday reviewed the sheets of Somerville's draft manuscript and sent the following response: (see also next note.)

"...I cannot resist saying too what pleasure I feel in your approbation of my later Experimental Researches. The approval of one judge is to me more stimulating than the applause of thousands that cannot understand the subject." Dep c.370, 20, MSF-1: Michael Faraday to Mary Somerville, 1 March 1834, Mary Somerville Collection, Bodleian Library, Oxford University.

In her first edition Somerville had adopted Faraday's ideas about electricity but continued to use an older vocabulary. Somerville revised the vocabulary in her second edition in 1835 and incorporated materials on Faraday's research in five topic areas: definite proportions, atomic weights, definite proportions of electricity (Faraday's laws of electricity), crystallization, and the density of the atmosphere. (Patterson, Elizabeth, *Mary Somerville and the Cultivation of Science, 1815-1840*, Mattinus Nijhoff Publishers, p.144, 1983)

² Mary Somerville's experimental work included a series of studies on the permeability of various bodies to the chemical rays of the sun. She carried out this work in her garden at Chelsea. The work involved an application of a type of primitive photography whereby light in passing through different media produced discolorations on paper coated with silver chloride. The media Somerville used included various coloured glasses, clear and coloured mica, emeralds, garnets, beryls, tourmalines, and rock crystal. On 12 October, 1835 Michael Faraday wrote to Somerville:

"I have been making some experiments with the papers but do not succeed in obtaining so good & regular a result as I wished & believed I might obtain.

In the first place the precipitates made upon the paper are not so sensible or regular as that first formed & washed & applied in the usual way, the excess of the muriate or nitrate used & the resulting salt formed interfering with action of light by retarding more or less the change and that in an irregular manner. Chloride produced on the paper is therefore nothing like so regular in its change as chloride previously precipitated & well washed.

In the next place I do not find that I can lay a more regular coat of the substance in the method I mentioned than by using the moist precipitated chloride & a camel hair pencil.

I suspect your chloride is a good deal [illegible]. I will therefore precipitate & wash some and send it to you in the moist state. Allow me to suggest that when you refer to and apply it to paper for your experiments you do so in a dark place, or by candlelight only & thus you may keep it for a long time in good condition.

I send also Biot's report for your inspection." Dep c.370, 20, MSF-1: Michael Faraday to Mary Somerville, 12 Oct. 1835, Mary Somerville Collection, Bodleian Library, Oxford University.

Somerville was investigating whether the "chemical" solar rays, which were known to blacken silver and fade vegetable colours, displayed analogous activity in passing through various media. The then accepted solar spectrum consisted of five overlapping spectra: calorific (i.e. infrared), red, yellow, blue, and chemical (i.e. ultraviolet). Somerville's experimental work always exhibited careful design, economy, attention to the control of

various parameters and the maintenance of standard controls (Patterson, Elizabeth, *Mary Somerville and the Cultivation of Science, 1815-1840*, Mattinus Nijhoff Publishers, p.173, 1983.)

³ *parallelopipedon*. A parallelopiped, *Webster's Revised Unabridged, 1913*.

⁴ Corrected from 1st edition which reads "...by article 75. The second member..."