BOOK I

CHAPTER V

THE MOTION OF A SOLID BODY OF ANY FORM WHATEVER

169. IF a solid body receives an impulse in a direction passing through its centre of gravity, all its parts will move with an equal velocity; but if the direction of the impulse passes on one side of that centre, the different parts of the body will have unequal velocities, and from this inequality results a motion of rotation in the body round its centre of gravity, at the same time that the centre is moved forward, or translated with the same velocity it would have taken, had the impulse passed through it. Thus the double motions of rotation and translation are produced by one impulse.

170. If a body rotates about its centre of gravity, or about an axis, and is at the same time carried forward in space; and if an equal and contrary impulse be given to the centre of gravity, so as to stop its progressive motion, the rotation will go on as before it received the impulse.

171. If a body revolves about a fixed axis, each of its particles will describe a circle, whose plane is perpendicular to that axis, and its radius is the distance of the particle from the axis. It is evident, that every point of the solid will describe an arc of the same number of degrees in the same time; hence, if the velocity of each particle be divided by its radius or distance from the axis, the quotient will be the same for every particle of the body. This is called the angular velocity of the solid.

172. The axis of rotation may change at every instant, the angular velocity is therefore the same for every particle of the solid for any one instant, but it may vary from one instant to another.

173. The general equations of the motion of a solid body are the same with those of a system of bodies, provided we assume the bodies m, m', m'', &c. to be a system of particles, infinite in number, and united into a solid mass by their mutual attraction.

Let x, y, z, be the co-ordinates of dm, a particle of a solid body urged by the forces X, Y, Z, parallel to the axes of the co-ordinates; then if **S** the sign of ordinary integrals[1] be put for Σ, and dm for m, the general equations of the motion of a system of bodies in article 158 become

$$\begin{aligned} \mathbf{S}\cdot\frac{d^2x}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{X}dm, \\ \mathbf{S}\cdot\frac{d^2y}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{Y}dm, \\ \mathbf{S}\cdot\frac{d^2z}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{Z}dm, \end{aligned} \qquad \textbf{(28)}$$

[and][2]

$$\mathbf{S}\cdot\left(\frac{xd^2y-yd^2x}{dt^2}\right)dm=\mathbf{S}\cdot\left(x\mathrm{Y}-y\mathrm{X}\right)dm,$$
$$\mathbf{S}\cdot\left(\frac{zd^2x-xd^2z}{dt^2}\right)dm=\mathbf{S}\cdot\left(\mathrm{z}\mathrm{X}-x\mathrm{Z}\right)dm, \qquad \mathbf{(29)}$$
$$\mathbf{S}\cdot\left(\frac{yd^2z-zd^2y}{dt^2}\right)dm=\mathbf{S}\cdot\left(y\mathrm{Z}-z\mathrm{Y}\right)dm,$$

which are the general equations of the motion of a solid, of which m is the mass.

Determination of the general Equations of the Motion of the Centre of Gravity of a Solid in Space

174. Let $\overline{x}+x'$, $\overline{y}+y'$, $\overline{z}+z'$ be put for *x*, *y*, *z*, in equations (28) then

$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{x}+d^2x'}{dt^2}\right\}=\mathbf{S}\cdot \mathrm{X}dm$$
$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{y}+d^2y'}{dt^2}\right\}=\mathbf{S}\cdot \mathrm{Y}dm \qquad \mathbf{(30)}$$
$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{z}+d^2z'}{dt^2}\right\}=\mathbf{S}\cdot \mathrm{Z}dm$$

in which $\overline{x}$, $\overline{y}$, $\overline{z}$, are the co-ordinates of *o* the moveable centre of gravity of the solid referred to P a fixed point, and x', y', z', are the co-ordinates of *dm* referred to *o*, fig. 47. Now the co-ordinates of the centre of gravity being the same for all the particles of the solid,

$$\mathbf{S}\,.\,dm\frac{d^2\overline{x}}{dt^2}=m\frac{d^2\overline{x}}{dt^2}$$
$$\mathbf{S}\,.\,dm\frac{d^2\overline{y}}{dt^2}=m\frac{d^2\overline{y}}{dt^2}$$
$$\mathbf{S}\,.\,dm\frac{d^2\overline{z}}{dt^2}=m\frac{d^2\overline{z}}{dt^2}.$$

And, with regard to the centre of gravity,

$$\mathbf{S}\,.\,x'dm=0$$
$$\mathbf{S}\,.\,y'dm=0$$
$$\mathbf{S}\,.\,z'dm=0$$

which denote the sum of the particles of the body into their respective distances from the origin; therefore their differentials are

$$\mathbf{S}.dm\frac{d^2x'}{dt^2}=0$$
$$\mathbf{S}.dm\frac{d^2y'}{dt^2}=0$$
$$\mathbf{S}.dm\frac{d^2z'}{dt^2}=0.$$

This reduces the equations (30) to

$$m\frac{d^2\overline{x}}{dt^2}=\mathbf{S}.\,\mathrm{X}dm$$
$$m\frac{d^2\overline{y}}{dt^2}=\mathbf{S}.\,\mathrm{Y}dm \qquad \mathbf{(31)}$$
$$m\frac{d^2\overline{z}}{dt^2}=\mathbf{S}.\,\mathrm{Z}dm.$$

These three equations determine the motion of the centre of gravity of the body in space, and are similar to those which give the motion of the centre of gravity of a system of bodies.

The solid therefore moves in space as if its mass were united in its centre of gravity, and all the forces that urge the body applied to that point.

175. If the same substitution be made in the first of equations (29), and if it be observed that $\overline{x}$, $\overline{y}$, $\overline{z}$, are the same for all the particles

$$\mathbf{S}\left(\overline{x}d^2\overline{y}-\overline{y}d^2\overline{x}\right)dm=m\left(\overline{x}d^2y-\overline{y}d^2\overline{x}\right)$$
$$\mathbf{S}\left(\overline{x}\mathrm{Y}-\overline{y}\mathrm{X}\right)dm=\overline{x}\cdot\mathbf{S}\cdot\mathrm{Y}dm-\overline{y}\cdot\mathbf{S}\cdot\mathrm{X}dm\,;$$

also

$$\mathbf{S}\left(x'd^2\overline{y}-y'd^2\overline{x}+\overline{x}d^2y'-\overline{y}d^2x'\right)dm=$$
$$d^2\overline{y}\cdot\mathbf{S}\cdot x'dm-d^2\overline{x}\cdot\mathbf{S}\cdot y'dm+\overline{x}\cdot\mathbf{S}\cdot d^2y'dm-\overline{y}\cdot\mathbf{S}\cdot d^2x'dm=0,$$

because x', y', z', are referred to the centre or gravity as the origin of the co-ordinates; consequently the co-ordinates $\overline{x}$, $\overline{y}$, $\overline{z}$, and their differentials vanish from the equation, which therefore retains its original form. Similar results will be obtained for the areas on the other two co-ordinate planes, and thus, equations (29) retain the same forms, whether the centre of gravity be in motion or at rest, proving the motions of rotation and translation to be independent of one another.

Rotation of a Solid

176. If to abridge

$$\mathbf{S}\left(y\mathrm{Z}-z\mathrm{Y}\right)dm=M,$$
$$\mathbf{S}\left(z\mathrm{X}-x\mathrm{Z}\right)dm=M',$$
$$\mathbf{S}\left(x\mathrm{Y}-y\mathrm{X}\right)dm=M''.$$

The integrals of equations (29), with regard to the time, will be

$$\begin{aligned}\mathbf{S}\left(\frac{ydz-zdy}{dt}\right)dm&=\int Mdt,\\ \mathbf{S}\left(\frac{zdx-xdz}{dt}\right)dm&=\int M'dt,\\ \mathbf{S}\left(\frac{xdy-ydx}{dt}\right)dm&=\int M''dt.\end{aligned}\qquad\textbf{(32)}$$

These equations, which express the properties of areas, determine the rotation of the solid;–equations (31) give the motion of its centre of gravity in space. **S** expresses the sum of the particles of the body, and $\int$ relates to the time alone.

177. Impetus is the mass into the square of the velocity, but the velocity of rotation depends on the distance from the axis, the angle being the same; hence the impetus of a revolving body is the sum of the products of each particle, multiplied by the square of its distance from the axis of rotation. Suppose oA, oB, oC, fig. 10, to be the co-ordinates of a particle dm, situate in m, and let them be represented by x, y, z; then because $m\mathrm{A}=\mathrm{R}o$, $m\mathrm{B}=\mathrm{Q}o$, $m\mathrm{C}=\mathrm{P}o$, the squares of the distances of dm from the three axes ox, oy, oz, are respectively

$$\left(m\mathrm{A}\right)^2=y^2+z^2,\quad\left(m\mathrm{B}\right)^2=x^2+z^2,\quad\left(m\mathrm{C}\right)^2=x^2+y^2.$$

Hence if A′, B′, C′, be the impetus or moments of inertia of a solid with regard to the axes ox, oy, oz, then

$$\begin{aligned}\mathrm{A}'&=\mathbf{S}\,.\,dm\left(y^2+z^2\right)\\ \mathrm{B}'&=\mathbf{S}\,.\,dm\left(x^2+z^2\right)\\ \mathrm{C}'&=\mathbf{S}\,.\,dm\left(x^2+y^2\right).\end{aligned}\qquad\textbf{(33)}$$

178. If an impulse be given to a sphere of uniform density, in a direction which does not pass through its centre of gravity, it will revolve about an axis perpendicular to the plane passing through the centre of the sphere and the direction of the force; and it will continue to rotate about the same axis even if new forces act on the sphere, provided they act equally on all its particles; and the areas which each of its particles describes will be constant.

179. If the solid be not a sphere, it may change its axis of rotation at every instant; it is therefore of importance, to ascertain if any axes exist in the solid, about which it would rotate permanently.

180. If a body rotates permanently about an axis, the rotatory pressures arising from the centrifugal forces of the solid are equal and contrary in each point of the axis, so that their sum is zero, and the areas described by every particle in the solid are proportional to the time; but if foreign forces disturb the rotation, the rotatory pressures on the axis of rotation are unequal, which causes a perpetual change of axis, and a variation in the areas described by the particles of the body, so that they are no longer proportional to the time. Thus the inconstancy of areas becomes a test of disturbing forces. In this disturbed rotation the body may be considered to have a permanent rotation during an instant only.

181. When three axes of a solid body are permanent axes of rotation, the rotatory pressures on them are zero; this is expressed by the equations

$$\mathbf{S}\,.\,xydm = 0;\;\; \mathbf{S}\,.\,xzdm = 0;\;\; \mathbf{S}\,.\,yzdm = 0;$$

which characterize such axes. To show this, it is necessary to prove that when these equations hold, the rotation of the body round any one axis causes no twisting effort to displace that axis; for example, that the centrifugal forces developed by rotation round *z*, produce no rotatory pressure round *y* and *x*; and so for the other, and *vice versâ*.

Demonstration. Let $r = \sqrt{x^2 + y^2}$ be the distance of a particle dm from z the axis of rotation, and let w be the angular velocity of the particle. By article 171 $w = \frac{v}{r}$, therefore $w^2\,.\,r = \frac{v^2}{r}$ is the œntrifugal force arising from rotation round z, and acting in the direction r. When resolved in the direction x, and multiplied by dm it gives

$$w^2 rdm \cdot \frac{x}{r} = w^2 xdm\,,$$

which, regarded as a force tending to turn the system round y, gives rotatory pressure = $w^2 xzdm$, because it acts at the distance z from the axis y. Therefore when $\mathbf{S}\,.\,xzdm = 0$, the whole effect is zero. Similarly, when $\mathbf{S}\,.\,yzdm = 0$, the whole effect of the revolving system to turn round x vanishes. Therefore, in order that z should be [the] permanent axis of rotation,

$$\mathbf{S}\,.\,xzdm = 0,\;\; \mathbf{S}\,.\,yzdm = 0.$$

In like manner, in order that y should be so,

$$\mathbf{S}\,.\,xydm = 0,\ \ \mathbf{S}\,.\,zydm = 0$$

must exist; and in order that x should be so,

$$\mathbf{S}\,.\,yxdm = 0,\ \ \mathbf{S}\,.\,zxdm = 0$$

must exist, all of which are in fact only three different equations, namely

$$\mathbf{S}\,.\,xydm = 0,\ \ \mathbf{S}\,.\,xzdm = 0,\ \ \mathbf{S}\,.\,yzdm = 0\,; \qquad \textbf{(34)}$$

and if these hold at once, x, y, z, will all be permanent axes of rotation.

Thus the impetus is as the square of the distance from the axis of rotation, and the rotatory pressures are simply as the distance from the same axis.

182. In order to ascertain whether a solid possesses any permanent axes of rotation, let the origin be a fixed point, and let x', y', z', be the co-ordinates of a particle dm, fixed in the solid, but revolving with it about its centre of gravity. The whole theory of rotation is derived from the equations (32) containing the principle of areas. These are the areas projected on the fixed co-ordinate planes xoy, xoz, yoz, fig. 48; but if ox', oy', oz', be the new axes that revolve with the solid, and if the values of x, y, z, given in article 163, be substituted, they will be the same sums, when projected on the new co-ordinate planes $x'oy'$, $x'oz'$, $y'oz'$. The angles q, y, and f, introduced by this change are arbitrary, so that the position of the new axes ox', oy', oz', in the solid, remains indeterminate; and these three angles may be made to fulfil any conditions of the problem.

183. The equations of rotation will take the most simple form if we suppose x', y', z', to be the principal axes of rotation, which they will become if the values of q, y, and f, can be so assumed as to make the rotatory pressures $\mathbf{S}\,.\,x'z'dm$, $\mathbf{S}\,.\,x'y'dm$, $\mathbf{S}\,.\,y'z'dm$, zero at once, then the equations (32) of the areas, when transformed to functions of x', y', z', and deprived of these terms, will determine the rotation of the body about its principal, or permanent axes of rotation, x', y', z'.

184. If the body has no principal axes of rotation, it will be impossible to obtain such values of q, f, and y, as will make the rotatory pressures zero; it must therefore be demonstrated whether or not it be possible to determine the angles in question, so as to fulfil the requisite condition.

185. To determine the existence and position of the principal axes of the body, or the angles q, f, and y, so that

$$\mathbf{S}\,.\,x'y'dm = 0;\ \ \mathbf{S}\,.\,x'z'dm = 0;\ \ \mathbf{S}\,.\,y'z'dm = 0\,.$$

Let values of x', y', z', in functions of x, y, z, determined from the equations in article 163 be substituted in the preceding expressions, then if to abridge,

$$\mathbf{S}.x^2dm = l^2;\quad \mathbf{S}.y^2dm = n^2;\quad \mathbf{S}.z^2dm = s^2$$

$$\mathbf{S}.xydm = f;\quad \mathbf{S}.xzdm = g;\quad \mathbf{S}.yzdm = h,$$

there will result

$$\begin{aligned}
&\cos f\,.\mathbf{S}.x'z'dm - \sin f\,.\mathbf{S}.y'z'dm = \\
&\left(l^2 - n^2\right)\sin q \sin y \cos y + f \sin q\left(\cos^2 y - \sin^2 y\right) \\
&+ \cos q\left(g\cos y - h\sin y\right); \\
&\sin f\,.\mathbf{S}.x'z'dm + \cos f\,.\mathbf{S}.y'z'dm = \\
&\sin q \cos q\left\{l^2 \sin^2 y + n^2 \cos^2 y - s^2 + 2f\sin y \cos y\right\} \\
&+\left(\cos^2 q - \sin^2 q\right).\left(g\sin y + h\cos y\right).
\end{aligned} \qquad \textbf{(35)}$$

If the second members of these be made equal to zero, there will be

$$\tan q = \frac{h\sin y - g\cos y}{\left(l^2 - n^2\right)\sin y \cos y + f\left(\cos^2 y - \sin^2 y\right)},$$

and

$$\tfrac{1}{2}\tan 2q = \frac{g\sin y + h\cos y}{s^2 - l^2\sin^2 y - n^2\cos^2 y - 2f\sin y\cos y},$$

but

$$\tfrac{1}{2}\tan 2q = \frac{\tan q}{1 - \tan^2 q},$$

by the arithmetic of sines; hence, equating these two values of $\tfrac{1}{2}\tan 2q$, and substituting for $\tan q$ its value in y; then if to abridge, $u = \tan y$, after some reduction it will be found that

$$\begin{aligned}
&0 = \left(gu + h\right)\left(hu - g\right)^2 + \\
&\left\{\left(l^2 - n^2\right)u + f\left(1 - u^2\right)\right\}.\left\{\left(hs^2 - hl^2 + fg\right)u + gn^2 - gs^2 - hf\right\};
\end{aligned}$$

where u is of the third degree. This equation having at least one real root, it is always possible to render the first members of the two equations (35) zero at the same time, and consequently

$$\left(\mathbf{S}.x'z'dm\right)^2 + \left(\mathbf{S}.y'z'dm\right)^2 = 0.$$

But that can only be the case when $\mathbf{S}.x'z'dm = 0$, $\mathbf{S}.y'z'dm = 0$. The value of $u = \tan y$, gives y, consequently $\tan q$ and q become known.

It yet remains to determine the condition $\Sigma . x'y'dm = 0$, and the angle φ. If substitution be made in $\Sigma . x'y'dm = 0$, for x' and y' from article 163, it will take the form $H \sin 2\varphi + L \cos 2\varphi$, H and L being functions of the known quantities θ and ψ; as it must be zero, it gives

$$\tan 2\varphi = -\frac{L}{H};$$

and thus the three axes ox', oy', oz', determined by the preceding values of θ, ψ, and φ, satisfy the equations

$$\Sigma . x'z'dm = 0, \ \Sigma . y'z'dm = 0, \ \Sigma . x'y'dm = 0 .$$

186. The equation of the third degree in u seems to give three systems of principal axes, one for each value of u; but u is the tangent of the angle formed by the axis x with the line of intersection of the plane xy with that of $x'y'$; and as any one of the three axes, x', y', z', may be changed into any other of them, since the preceding equations will still be satisfied, therefore the equation in u will determine the tangent of the angle formed by the axis x with the line of intersection of xy and $x'y'$, with that of xy and $x'z'$, or with that of xy and $y'z'$. Consequently the three roots of the equation in u are real, and belong to the same system of axes.

187. Whence every body has at least one system of principal and rectangular axes, round any one of which if the body rotates, the opposite centrifugal forces balance each other. This theorem was first proposed by Segner[3] in the year 1755, and was demonstrated by Albert Euler[4] in 1760.

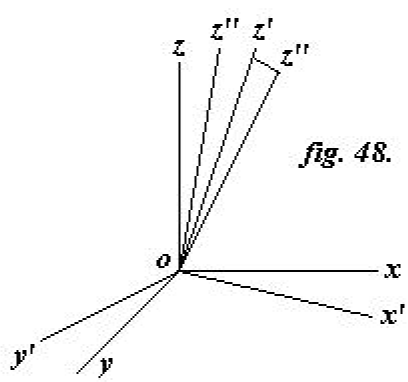


fig. 48.

188. The position of the principle axes ox', oy', oz', in the interior of the solid, is now completely fixed; and if there be no disturbing forces, the body will rotate permanently about any one of them, as oz', fig. 48; but if the rotation be disturbed by foreign forces, the solid will only rotate for in instant about oz', and in the next element of time it will rotate about oz'', and so on, perpetually changing. Six equations are therefore required to fix the position of the instantaneous axis oz''; three will determine its place with regard to the principal axes ox', oy', oz', and three more are necessary to determine the position of the principal axes themselves in space, that is, with regard to the fixed co-ordinates *ox*, *oy*, *oz*. The permanency of rotation is not the same for all the three axes, as will now be shown.

189. The principal axes possess this property—that the moment of inertia of the solid is a maximum for one of these, and a minimum for another. Let x', y', z', be the co-ordinates of *dm*, relative to the three principal axes, and let x, y, z, be the co-ordinates of the same element referred to any axes whatever having the same origin. Now if

$$C' = \mathbf{S}\left(x^2 + y^2\right)dm$$

be the moment of inertia relatively to one of these new axes, as *z*, then substituting for *x* and *y* their values from article 163, and making

$$A = \mathbf{S}\left(y'^2 + z'^2\right).dm;\ \ B = \mathbf{S}\left(x'^2 + z'^2\right).dm;\ \ C = \mathbf{S}\left(x'^2 + y'^2\right).dm;$$

the value of C' will become

$$C' = A\sin^2 q\sin^2 f + B\sin^2 q\cos^2 f + C\cos^2 q,$$

in which

$$\sin^2 q\sin^2 f,\ \sin^2 q\cos^2 f,\ \cos^2 q,$$

are the squares of the cosines of the angles made by ox', oy', oz', with *oz*; and *A*, *B*, *C*, are the moments of inertia of the solid with regard to the axes x', y', and z', respectively. The quantity C' is less than the greatest of the three quantities *A*, *B*, *C*, and exceeds the least of them; the greatest and the least moments of inertia belong therefore, to the principal axes. In fact, C' must be less than the greatest of the three quantities *A*, *B*, *C*, because their joint coefficients are always equal to unity; and for a similar reason it is always greater than the least.

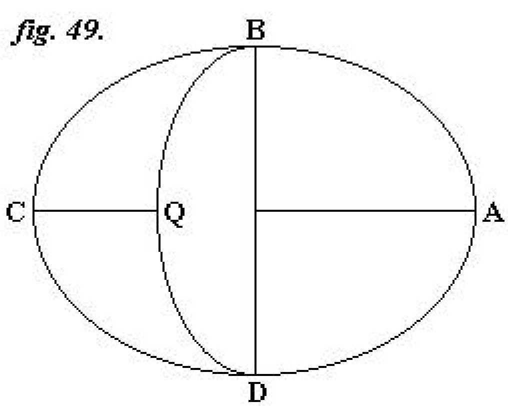
190. When *A* = *B* = *C*, then all the axes of the solid are principal axes, and it will rotate permanently about any one of them. The sphere of uniform density is a solid of this kind, but there are many others.

191. When two of the moments of inertia are equal, as *A*=*B*, then

$$C' = A\sin^2 q + C\cos^2 q\ ;$$

and all the moments of inertia in the same plane with these are equal: hence all the axes situate in that plane are principal axes. The ellipsoid of revolution of uniform density is of this kind; all the axes in the plane of its equator being principal axes.

fig. 49.



192. An ellipsoid of revolution is formed by the rotation of an ellipse ABCD about its minor axis BD. Then AC is its equator. When the moments of inertia are unequal, the rotation round the axes which have their moment of inertia a maximum or minimum is stable, that is, round the least or greatest axis; but the rotation is unstable round the third, and may be destroyed by the slightest cause. If stable rotation be slightly deranged, the body will never deviate far from its equilibrium; whereas in unstable rotation, if it be disturbed, it will deviate more and more, and will never return to its former state.

193. This theorem is chiefly of importance with regard to the rotation of the earth. If xoy (fig. 46) be the plane of the ecliptic, and z its pole; $x'oy'$ the plane of the equator, and z' its pole: then oz' is the axis of the earth's rotation, $zoz' = q$ is the obliquity of the ecliptic, gN the line of the equinoxes, and g the first point of Aries: hence $xog = y$ is the longitude of ox, and $x'og = f$ is the longitude of the principal revolving axis ox', or the measure of the earth's rotation: oz' is therefore one of the permanent axes of rotation.

The earth is flattened at the poles, therefore oz' is the least of the permanent axes of rotation, and the moment of inertia with regard to it, is a maximum. Were there no disturbing forces, the earth would rotate permanently about it; but the sun and moon, acting unequally on the different particles, disturb its rotation. These disturbing forces do not sensibly alter the velocity of rotation, in which neither theory nor observation have detected any appreciable variation; nor do they sensibly displace the poles of rotation on the surface of the earth; that is to say, the axis of rotation, and the plane of the equator which is perpendicular to it, always meet the surface in the same points; but these forces alter the direction of the polar axis in space, and produce the phenomena of precession and nutation;[5] for the earth rotates about oz'', fig. 50, while oz'' revolves about its mean place oz', and at the same time oz' describes a cone about oz; so that the motion of the axis of rotation is very complicated. That axis of rotation, of which all the points remain at rest during the time dt, is called an instantaneous axis of rotation, for the solid revolves about it during that short interval, as it would do about a fixed axis.

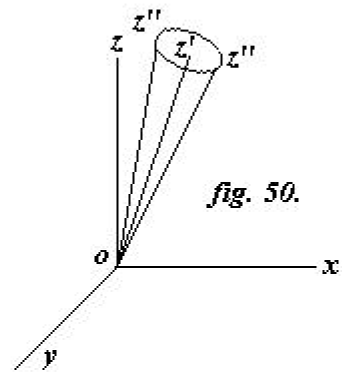


fig. 50.

The equations (32) must now be so transformed as to give all the circumstances of rotatory motion.

194. The equations in article 163, for changing the co-ordinates, will become

$$\begin{aligned} x &= ax' + by' + cz' \\ y &= a'x' + b'y' + c'z' \\ z &= a''x' + b''y' + c''z' \end{aligned} \qquad \textbf{(36)}$$

If to abridge

$$\begin{aligned} a &= \cos q \sin y \sin f + \cos y \cos f \\ b &= \cos q \sin y \cos f - \cos y \sin f \\ c &= \sin q \sin y \\ a' &= \cos q \cos y \sin f - \sin y \cos f \\ b' &= \cos q \cos y \cos f + \sin y \sin f \\ c' &= \sin q \cos y \\ a'' &= -\sin q \sin f \\ b'' &= -\sin q \cos f \\ c'' &= \cos q, \end{aligned}$$

where a, b, c are the cosines of the angles made by x with x', y', z'; a', b', c' are the cosines of the angles made by y with x', y', z'; and a'', b'', c'' are the cosines of the angles made by z with the same axes.

Whatever the co-ordinates of dm may be, since they have the same origin,

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 .$$

By means of these, it may be found that

$$\begin{aligned} a^2 + a'^2 + a''^2 &= 1 \qquad & ab + a'b' + a''b'' &= 0 \\ b^2 + b'^2 + b''^2 &= 1 \qquad & ac + a'c' + a''c'' &= 0 \\ c^2 + c'^2 + c''^2 &= 1 \qquad & bc + b'c' + b''c'' &= 0 . \end{aligned}$$

In the same manner, to obtain x', y', z', in functions of x, y, z,

$$\begin{aligned} x' &= ax + a'y + a''z \\ y' &= bx + b'y + b''z \\ z' &= cx + c'y + c''z , \end{aligned} \qquad \textbf{(37)}$$

whence the equations of condition,

$$\begin{aligned} a^2 + b^2 + c^2 &= 1 \qquad & aa' + bb' + cc' &= 0 \\ a'^2 + b'^2 + c'^2 &= 1 \qquad & aa'' + bb'' + cc'' &= 0 \\ a''^2 + b''^2 + c''^2 &= 1 \qquad & a'a'' + b'b'' + c'c'' &= 0 \end{aligned}$$

six of the quantities a, b, c, a', b', c', a'', b'', c'', are determined by the preceding equations, and three remain arbitrary.

If values of x', y', z', found from equations (36) be compared with their values in equations (37), there will result

$$\begin{aligned} a &= b'c'' - b''c' \qquad & a' &= b''c - bc'' \qquad & a'' &= bc' - b'c \\ b &= a''c' - a'c'' \qquad & b' &= ac'' - a''c \qquad & b'' &= a'c - ac' \\ c &= a'b'' - a''b' \qquad & c' &= a''b - ab'' \qquad & c'' &= ab' - a'b. \end{aligned} \qquad \textbf{(38)}$$

195. The axes x', y', z', retain the same position in the interior of the body during its rotation, and are therefore independent of the time; but the angles a, b, c, a', b', c', a'', b'', c'', vary with the time; hence, if values of y, z, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from equations (36), be substituted in the first of equations (32), it will become

$$\mathbf{S}\left\{\begin{array}{c}\left(\frac{a'da''-a''da'}{dt}\right)x'^2+\left(\frac{b'db''-b''db'}{dt}\right)y'^2+\left(\frac{c'dc''-c''dc'}{dt}\right)z'^2\\ +\left(\frac{a'db''-b''da'+b'da''-a''db'}{dt}\right)x'y'\\ +\left(\frac{a'dc''-c''da'+c'da''-a''dc'}{dt}\right)x'z'\\ +\left(\frac{b'dc''-c''db'+c'db''-b''dc'}{dt}\right)y'z'\end{array}\right\}dm=\int M\cdot dt.$$

If a', a'', b', &c. be eliminated from this equation by their values in (38), and if to abridge

$$\begin{aligned} cdb+c'db'+c''db'' &= -bdc-b'dc'-b''dc'' = pdt\\ adc+a'dc'+a''dc'' &= -cda-c'da'-c''da'' = qdt \qquad \textbf{(39)}\\ bda+b'da'+b''da'' &= -adb-a'db'-a''db'' = rdt \end{aligned}$$

$$A=\mathbf{S}\left(y'^2+z'^2\right)dm;\;\; B=\mathbf{S}\left(x'^2+z'^2\right)dm;\;\; C=\mathbf{S}\left(x'^2+y'^2\right)dm.$$

And if

$$\mathbf{S}\,.\,x'y'dm=0 \quad \mathbf{S}\,.\,x'zdm=0 \quad \mathbf{S}\,.\,y'z'dm=0\,,$$

it will be found that

$$aAp+bBq+cCr=\int Mdt\,;$$

by the same process it may be found that

$$a'Ap+b'Bq+c'Cr=\int M'dt\,,$$
$$a''Ap+b''Bq+c''Cr=\int M''dt\,.$$

196. If the differentials of these three equations be taken, making all the quantities vary except A, B, and C, then the sum of the first differential multiplied by a, plus the second multiplied by a', plus the third multiplied by a'', will be

$$A\frac{dp}{dt}+\left(C-B\right)\cdot qr=aM+a'M'+a''M''\,,$$

in consequence of the preceding relations between $a\ a'\ a''$, $b\ b'\ b''$, $c\ c'\ c''$, and their differentials. By a similar process the coefficient $b\ b'\ b''$, &c., may be made to vanish, and then if

$$\begin{aligned} aM+a'M'+a''M'' &= N\\ bM+b'M'+b''M'' &= N'\\ cM+c'M'+c''M'' &= N'' \end{aligned}$$

the equations in question are transformed to

$$A\frac{dp}{dt}+\left(C-B\right).qr=N$$
$$B\frac{dq}{dt}+\left(A-C\right).rp=N'$$
$$C\frac{dr}{dt}+\left(B-A\right).pq=N''. \qquad \textbf{(40)}$$

And if $a,\ a',\ a'',\ b,\ b'$, &c., and their differentials, be replaced by their functions in $f,\ y$, and y, given in article 194, the equations (39) become

$$pdt=\sin f\sin q\ .dy\ -\cos f\ .dq$$
$$qdt=\cos f\sin q\ .dy\ +\sin f\ .dq \qquad \textbf{(41)}$$
$$rdt=df-\cos q\ .dy\ .$$

197. These six equations contain the whole theory of the rotation of the planets and their satellites, and as they have been determined in the hypothesis of the rotatory pressures being zero, they will give their rotation nearly about their principal axes.

198. The quantities $p,\ q,\ r$, determine oz'', the position of the real and instantaneous axis of rotation, with regard to its principal axis oz'; when a body has no motion but that of rotation, all the points in a permanent axis of rotation remain at rest; but in an instantaneous axis of rotation the axis can only be regarded as at rest from one instant to another.

If the equations (36) for changing the co-ordinates, be resumed, then with regard to the axis of rotation,

$$dx = 0,\ dy = 0,\ dz = 0,$$

since all its points are at rest; therefore the indefinitely small spaces moved over by that axis in the direction of these co-ordinates being zero, the equations in question become,

$$x'da\ +\ y'db\ +\ z'dc\ =\ 0,$$
$$x'da'\ +\ y'db'\ +\ z'dc'\ =\ 0,$$
$$x'da''\ +\ y'db''\ +\ z'dc''=\ 0,$$

which will determine $x',\ y',\ z'$, and consequently oz'' the axis in question.

For if the first of these equations be multiplied by c, the second by c', and the third by c'', their sum is

$$py'-qz'=0\ . \qquad \textbf{(42)}$$

Again, if the first be multiplied by b, the second by b', and the third by b'', their sum is[6]

$$rx' - pz' = 0. \tag{43}$$

Lastly, if the first equation be multiplied by a, the second by a', and the third by a'', their sum is

$$qz' - ry' = 0.$$

The last of these is contained in the two first, which are the equations to a straight line oz'', which forms, with the principle axes x', y', z', angles whose cosines are

$$\frac{p}{\sqrt{p^2+q^2+r^2}}; \quad \frac{q}{\sqrt{p^2+q^2+r^2}}; \quad \frac{r}{\sqrt{p^2+q^2+r^2}}; \tag{44}$$

for the two last give

$$x'^2 = z'^2\frac{p^2}{r^2}; \quad y'^2 = z'^2\frac{q^2}{r^2};$$

whence

$$x'^2 + y'^2 + z'^2 = z'^2\left\{\frac{p^2+q^2+r^2}{r^2}\right\};$$

and therefore

$$\frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{r}{\sqrt{p^2+q^2+r^2}}.$$

But

$$oz'' = \sqrt{x'^2+y'^2+z'^2};$$

and

$$oz'' : oc :: 1 : \cos z''oc;$$

then if x', y', z', be the co-ordinates of the point z'',

$$\cos z''oc = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{r}{\sqrt{p^2+q^2+r^2}}.$$

In the same manner

$$\cos z''ox' = \frac{p}{\sqrt{p^2+q^2+r^2}}$$

and

$$\cos z''oy' = \frac{q}{\sqrt{p^2+q^2+r^2}}.$$

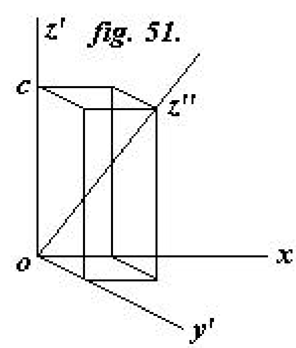


fig. 51.

Consequently oz'' is the instantaneous axis of rotation.

201.[7] The angular velocity of rotation is also given by these quantities. If the object be to determine it for a point in the axis, as for example where $oc = 1$, then

$$x' = 0,\ y' = 0,$$

and the values of dx, dy, dz give, when divided by dt,

$$\frac{dy}{dt}\sin q,\ \frac{dq}{dt}\cos q,\ -\frac{dq}{dt}\sin q,$$

for the components of the velocity of a particle; hence the resulting velocity is

$$\frac{\sqrt{dq^2 + dy^2 \sin^2 q}}{dt} = \sqrt{q^2 + r^2},$$

which is the sum of the squares of the two last of equations (41).

199. But in order to obtain the angular velocity of the body, this quantity must be divided by the distance of the particle at c' from the axis oz''; but this distance is evidently equal to the sine of $z''oc$, the angle between oz' and oz'', the principal and instantaneous axes of rotation; but

$$\frac{r}{\sqrt{p^2 + q^2 + r^2}}$$

is the cosine of this angle; hence

$$\sqrt{1 - \frac{r^2}{p^2 + q^2 + r^2}},$$

or

$$\frac{\sqrt{q^2 + p^2}}{\sqrt{p^2 + q^2 + r^2}},$$

is the sine; and therefore

$$\sqrt{p^2 + q^2 + r^2}$$

is the angular velocity of rotation. Thus, whatever may be the rotation of a body about a point that is fixed, or one considered to be fixed, the motion can only be rotation about an axis that is fixed during an instant, but may vary from one instant to another.

200. The position of the instantaneous axis with regard to the three principal axes, and the angular velocity of rotation, depend on p, q, r, whose determination is very important in these researches; and as they express quantities independent of the situation of the fixed plane xoy, they are themselves independent of it.

201.[8] Equations (40) determine the rotation of a solid troubled by the action of foreign forces, as for example, that of the earth when disturbed by the sun and moon. But the same equations will also determine the rotation of a solid, when not disturbed in its rotation.

Rotation of a Solid not subject to the action of Disturbing Forces, and at liberty to revolve freely about a Fixed Point, being its Centre of Gravity, or not

202. Values of *p*, *q*, *r*, in terms of the time must be obtained, in order to ascertain all the circumstances of rotation at every instant.

If we suppose that there are no disturbing forces, the areas are constant: hence the equations (40) become

$$\begin{aligned} A\cdot dp+(C-B)\cdot q\cdot r\cdot dt&=0;\\ B\cdot dq+(A-C)\cdot r\cdot p\cdot dt&=0;\\ C\cdot dr+(B-A)\cdot p\cdot q\cdot dt&=0. \end{aligned} \qquad \textbf{(45)}$$

If the first be multiplied by *p*, the second by *q*, and the third by *r*, their sum is

$$Apdp+Bqdq+Crdr=0,$$

and its integral is

$$Ap^2+Bq^2+Cr^2=k^2, \qquad \textbf{(46)}$$

k^2 being a constant quantity. Again, if the three equations be multiplied respectively by *Ap*, *Bq*, *Cr*, and integrated, they give

$$A^2p^2+B^2q^2+C^2r^2=h^2, \qquad \textbf{(47)}$$

a constant quantity. Equation (46)[9] contains the principle of the preservation of impetus or living force which is constant in conformity with article 148. From these two integrals are obtained:

$$\begin{aligned} p^2&=\frac{h^2-Bk+(B-C).Cr^2}{A(A-B)}\\ q^2&=\frac{h^2-Ak+(A-C).Cr^2}{B(B-A)}. \end{aligned} \qquad \textbf{(48)}$$

By the substitution of these values of *p* and *q*, the last of equations (45) when resolved according to *dt*, gives

$$dt=\frac{Cdr.\sqrt{AB}}{\sqrt{\left\{\left(h^2-Bk+(B-C)Cr^2\right).\left(-h^2+Ak+(C-A)\right).Cr^2\right\}}} \qquad \textbf{(49)}$$

This equation will give by quadratures the value of t in r, and reciprocally the value of r in t; and thus by the substitution of this value of r in equations (48) the three quantities p, q and r become known in functions of the time. This equation can only be integrated when any two of the moments of inertia are equal, either when

$$A = B, \quad B = C, \quad A = C;$$

in each of these cases the solid is a spheroid of revolution.

203. Thus p, q, r, being known functions of the time, the angular velocity of the solid, and its rotation with regard to the principal axes, are known at every instant.

204. This however is not sufficient. To become acquainted with all the circumstances of rotation, it is requisite to know the position of the principle axes themselves with regard to quiescent space, that is, their position relatively to the fixed axes x, y, z. But for that purpose the angles f, y, and q, must be determined in functions of the time, or, which is the same thing, in functions of p, q, r, which may now be regarded as known quantities.

If the first of equations (45) be multiplied by a, the second by b, and the third by c, their sum when integrated, in consequence of the relations between the angles in article 194, is

$$aAp + bBq + cCr = l,$$

by a similar process

$$\begin{aligned} a'Ap + b'Bq + c'Cr &= l', \\ a''Ap + b''Bq + c''Cr &= l'', \end{aligned} \qquad \textbf{(50)}$$

l, l', l'', being arbitrary constant quantities. These equations coincide with those in article 195, and contain the principle of areas. They are not however three distinct integrals, for the sum of their squares is

$$A^2p^2 + B^2q^2 + C^2r^2 = l^2 + l'^2 + l''^2,$$

in consequence of the equations in article 194. But this is the same with (47); hence

$$l^2 + l'^2 + l''^2 = h^2$$

being an equation of condition, equations (50) will only give values of two of the angles f, y, and q.

The constant quantities l, l', l'', correspond with c, c', c'', in article 164, therefore

$$\tfrac{1}{2}t\sqrt{l^2 + l'^2 + l''^2}$$

is the sum of the areas described in the time t by the projection of each particle of the body on the plane on which that sum is a maximum. If xoy be that plane, l and l' are zero: therefore, in every solid body in rotation about an axis, there exists a plane, on which the sum of the areas described by the projections of the particles of the solid during a finite time is a maximum. It is called the Invariable Plane, because it remains parallel to itself during the motion of the body: it is also named the plane of the Greatest Rotatory Pressure.

Since

$$l = 0, \quad l' = 0, \quad l'' = h,$$

if the first of equations (50) be multiplied by a, the second by a', and the third by a'', in consequence of the equations in article 194, their sum is

$$a'' = \frac{Ap}{h};$$

in the same manner it will be found that

$$b'' = \frac{Bq}{h}, \quad c'' = \frac{Cr}{h};$$

or, substituting the values of a'', b'', c'', from article 194,

$$\sin q' \sin f' = -\frac{Ap}{h}, \quad \sin q' \cos f' = -\frac{Bq}{h}, \quad \cos q' = \frac{Cr}{h}. \qquad \textbf{(51)}$$

The accented angles q', f', y', relate to the invariable plane, and angles q, f, y, to the fixed plane.

Because p, q, r, are known functions of the time, f' and q' are determined, and if dq be eliminated between the two first of equation (41), the result will be

$$\sin^2 q' \, . \, dy' = \sin q' \, . \sin f' \, . \, pdt + \sin q' \, . \cos f' \, . \, qdt \, .$$

But in consequence of equations (51), and because

$$Ap^2 + Bq^2 = k - Cr^2,$$

$$dy' = \frac{Cr^2 - k}{h^2 - C^2 r^2} \cdot hdt \, ;$$

and as r is given in functions of the time by equation (49), y' is determined.

Thus, p, q, r, y', q', and f', are given in terms of the time: so that the position of the three principal axes with regard to the fixed axes, ox, oy, oz; and the angular velocity of the body, are known at every instant.

205. As there are six integrations, there must be six arbitrary constant quantities for the complete solution of the problem. Besides h and k, two more will be introduced by the integration of dt and $d\psi'$. Hence two are still required, because by the assumption of *xoy* for the invariable plane, l and l' become zero.

Now the three angles, ψ', ϕ', θ', are given in terms of p, q, r, and these last are known in terms of the time; hence ψ', ϕ', θ', (fig. 49), are known with regard to the invariable plane *xoy*: and by trigonometry it will be easy to determine values of ψ, ϕ, θ, with regard to any fixed plane whatever, which will introduce two new arbitrary quantities, making in all six, which are requisite for the complete solution of the problem.

206. These two new arbitrary quantities are the inclination of the invariable plane on the fixed plane in question, and the angular distance of the line of intersection of these two planes from a line arbitrarily assumed on the fixed plane; and as the initial position of the fixed plane is supposed to be given, the two arbitrary quantities are known.

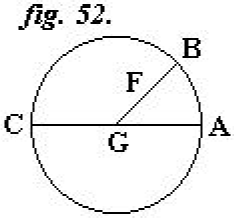
If the position of the three principal axes with regard to the invariable plane be known at the origin of the motion, ϕ', θ', will be given, and therefore p, q, r, will be known at that time; and then equation (46) will give the value of k.

The constant quantity arising from the integration of dt depends on the arbitrary origin or instant whence the time is estimated, and that introduced by the integration of $d\psi'$ depends on the origin of the angle ψ', which may be assumed at pleasure on the invariable plane.

207. The determination of the sixth constant quantity h is very interesting, as it affords the means of ascertaining the point in which the sun and planets may be supposed to have received a primitive impulse, capable of communicating to them at once their rectilinear and rotatory motions.

The sum of the areas described round the centre of gravity of the solid by the radius of each particle projected on a fixed plane, and respectively multiplied by the particles, is proportional to the moment of the primitive force projected on the same plane; but this moment is a maximum relatively to the plane which passes through the point of primitive impulse and centre of gravity, hence it is the invariable plane.

fig. 52.



208. Let G, fig. 52, be the centre of gravity of a body of which ABC is a section, and suppose that it has received an impulse in the plane ABC at the distance GF, from its centre of gravity; it will move forward in space at the same time that it will rotate about an axis perpendicular to the plane ABC. Let v be the velocity generated in the centre of gravity by the primitive impulse; then if m be the mass of the body, $m \cdot v \cdot \mathrm{GF}$ will be the moment of this impulse, and multiplying it by $\frac{1}{2}t$, the product will be equal to the sum of the areas described during the time t; but this sum was shown to be

$$\tfrac{1}{2}t\sqrt{l^2 + l'^2 + l''^2}\ ;$$

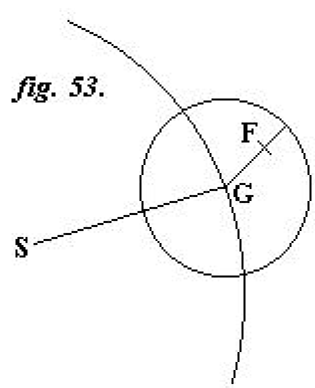
hence

$$\sqrt{l^2 + l'^2 + l''^2} = m \cdot v \cdot \mathrm{GF} = h\,;$$

which determines the sixth arbitrary constant quantity h. Were the angular velocity of rotation, the mass of the body and the velocity of its centre of gravity known, the distance GF, the point of primitive impulse, might be determined.

209. It is not probable that the primitive impulse of the planets and other bodies of the system passed exactly through their centres of gravity; most of them are observed to have a rotatory motion, though in others it has not been ascertained, on account of their immense distances, and the smallness of their volumes. As the sun rotates about an axis, he must have received a primitive impulse not passing through his centre of gravity, and therefore it would cause him to move forward in space accompanied by the planetary system, unless an impulse in the contrary direction had destroyed that motion, which is by no means likely. Thus the sun's rotation leads us to presume that the solar system may be in motion.

210. Suppose a planet of uniform density, whose radius is R, to be a sphere revolving round the sun in S, at the distance SG or $\bar{r}$, with an angular velocity represented by u, then the velocity of the centre of gravity will be $v = u\bar{r}$.



Imagine the planet to be put in motion by a primitive impulse, passing through the point F, fig. 53, then the sphere will rotate about an axis perpendicular to the invariable plane, with an angular velocity equal to r, for the components q and p at right angles to that plane will be zero; hence, the equation[10]

$$\sqrt{l^2 + l'^2 + l''^2} = m \cdot v \cdot \mathrm{GF}$$

becomes

$$l'' = mu\bar{r} \,.\, GF\,;$$

and

$$l'' = rC\,.$$

The centre of gyration is that point of a body in rotation, into which, if all the particles were condensed, it would retain the same degree of rotatory power. It is found that the square of the radius of gyration in a sphere, is equal to $\frac{2}{5}$ of the square of its semi-diameter; hence the rotatory inertia C becomes $\frac{2}{5}m\mathrm{R}^2$, thus

$$l'' = r \times \frac{2}{5} m\mathrm{R}^2, \text{ and } \mathrm{GF} = \frac{2}{5} \cdot \frac{\mathrm{R}^2}{\bar{r}} \cdot \frac{r}{u}.$$

211. Hence, if the ratio of the mean radius of a planet to its mean distance from the sun, and the ratio of its angular velocity of rotation to its angular velocity in its orbit, could be ascertained, the point in which the primitive impulse was given, producing its motion in space, might be determined.

212. Were the earth a sphere of uniform density, the ratio $\frac{\mathrm{R}}{r}$ would be 0.000042665; and the ratio of its rotatory velocity to that in its orbit is known by observation to be 366.25638, whence $\text{GF}=\frac{\text{R}}{160}$; and as the mean radius of the earth is about 4,000 miles, the primitive impulse must have been given at the distance of 25 miles from the centre. However, as the density of the earth is not uniform, but decreases from the centre to the surface, the distance of the primitive impulse from its centre of gravity must have been something less.

213. The rotation of the earth has established a relation between time and the arcs of a circle. Every point in the surface of the earth passes through 360^0 in 24 hours; and as the rotation is uniform, the arcs described are proportional to the time, so that one of these quantities may represent the other. Thus, if a be an arc of any number of degrees, and t the time employed to describe it, $360^0 : a :: 24 : t$: hence $a=\frac{360}{24}t$; or, if the constant co-efficient $\frac{360}{24}$ be represented by n, $a=nt$, and $\sin a=\sin nt$, $\cos a=\cos nt$.

In the same manner the periodic time of the moon being 27.3 days nearly, an arc of the moon's orbit would be $\frac{360}{27.3}t$, and may also be expressed by nt. Thus, n may have all values, so that nt is a general expression for any arc that increases uniformly with the time.

214. The motions of the planets are determined by equations of these forms,

$$\frac{d^2u}{dt^2}+n^2u=\text{R}$$
$$\frac{d^2u}{dt^2}+n^2u=0,$$

which are only transformations of the general equation of the motions of a system of bodies. The integrals of both give a value of u in terms of the sines and cosines of circular arcs increasing with the time; the first by approximation, but the integral of the second will be obtained by making $u=c^x$, c being the number whose Napierian[11] logarithm is unity. Whence

$$d^2u=c^x\left(d^2x+dx^2\right),$$

and the equation in question becomes

$$d^2x+dx^2+n^2dt^2=0.$$

Let

$$dx=ydt\text{, then } d^2x=dydt,$$

since the element of the time is constant, which changes the equation to

$$dy+dt\left(n^2+y^2\right)=0.$$

If $y = m$ a constant quantity, $dm = dy = 0$, hence

$$n^2 + m^2 = 0;$$

whence

$$m = \pm n\sqrt{-1},$$

but

$$dx = ydt = \pm ndt\sqrt{-1},$$

the integral of which is

$$x = \pm nt\sqrt{-1}.$$

As x has two values, $u = c^x$ gives

$$u = bc^{nt\sqrt{-1}}, \text{ and } u = b'c^{-nt\sqrt{-1}},$$

and because either of these satisfies the conditions of the problem, their sum

$$u = bc^{nt\sqrt{-1}} + u = b'c^{-nt\sqrt{-1}},$$

also satisfies the conditions and is the general solution, b and b' being arbitrary constant quantities. But

$$c^{nt\sqrt{-1}} = \cos nt + \sqrt{-1}\sin nt,$$
$$c^{-nt\sqrt{-1}} = \cos nt - \sqrt{-1}\sin nt.$$

Hence

$$u = (b + b')\cos nt + (b - b')\sqrt{-1}\sin nt.$$

Let

$$b + b' = M \sin e; \quad (b - b')\sqrt{-1} = M \cos e;$$

and then[12]

$$u = M\{\sin e \cos nt + \cos e \sin nt\}$$

or

$$u = M \sin(nt + e),$$

which is the integral required, because M and e are two arbitrary constant quantities.

215. Since a sine or cosine never can exceed the radius, $\sin.(nt + e)$ never can exceed unity, however much the time may increase; therefore u is a periodic quantity whose value oscillates between fixed limits which it never can surpass. But that would not be the case were n an imaginary quantity; for let

$$n = a \pm b\sqrt{-1};$$

then the two values of x become

$$x = bt + at\sqrt{-1} \qquad x = bt - at\sqrt{-1},$$

consequently,

$$c^{bt+at\sqrt{-1}} = c^{bt}\,.\,c^{at\sqrt{-1}} = c^{bt}\left\{\cos at + \sqrt{-1}\sin at\right\}$$
$$c^{bt-at\sqrt{-1}} = c^{bt}\,.\,c^{-at\sqrt{-1}} = c^{bt}\left\{\cos at - \sqrt{-1}\sin at\right\}$$

whence

$$u = c^{bt}\left\{(b+b')\cos at + (b-b')\sqrt{-1}\sin at\right\}$$

or substituting for

$$b+b'; \quad (b-b')\sqrt{-1};$$

[then][13]

$$u = c^{bt}\cdot M\cdot\sin(at+e).$$

But[14]

$$c^{bt} = 1 + bt + \tfrac{1}{2}b^2t^2 + \frac{1}{2.3}b^3t^3 + \&c.,$$

therefore c^{bt} increases indefinitely with the time, and u is no longer a periodic function, but would increase to infinity.

Were the roots of n^2 equal, then $x = bt$, and $u = \mathrm{C}.c^{bt}$, C being constant.

Thus it appears that if the roots of n^2 be imaginary or equal, the function u would increase without limit.

These circumstances are of the highest importance, because the stability of the solar system depends upon them.

Rotation of a Solid which turns nearly round one of its principal Axes, as the Earth and the Planets, but not subject to the action of accelerating Forces

216. Since the axis of rotation oz'' is very near oz', fig. 50, the angle $z'oz''$ is so small, that its cosine $\dfrac{r}{\sqrt{p^2+q^2+r^2}}$ differs but little from unity; hence p and q are so minute that their product may be omitted, which reduces equations (45) to

$$Cdr = 0,$$
$$Adp + (C-B)qrdt = 0,$$
$$Bdq + (A-C)prdt = 0;$$

the first shows the angular velocity to be uniform, and the two last give

$$\frac{d^2q}{dt^2}+\frac{(A-C)}{B}r\frac{dp}{dt}=0;\quad \frac{dp}{dt}=\frac{(B-C)}{A}qr=0;$$

hence if the constant quantity

$$\frac{(A-C)(B-C)}{AB}r^2=n^2,$$

the result will be

$$\frac{d^2q}{dt^2}+n^2q=0;$$

and by article 214,

$$q=M'\cos(nt+g).$$

In the same manner

$$p=M\sin(nt+g);$$

whence

$$M'=M\cdot\sqrt{\frac{A(A-C)}{B(B-C)}}.$$

217. If oz'' the real axis of rotation coincides with oz', the principal axis in the beginning of the motion, then q and p are zero; hence also, $M=0$, and $M'=0$. It follows therefore, that in this case q and p will always be zero, and the axis oz'' will always coincide with oz'; whence, if the body begins to turn round one of its principal axes, it will continue to rotate uniformly about that axis for ever. On account of this remarkable property these are called the natural axes of rotation; it belongs to them exclusively, for if the position of the real axis of rotation oz'' be invariable on the surface of the body, the angular velocity will be constant; hence

$$dp=0,\quad dq=0,\quad dr=0,$$

and

$$(C-B)\,qrdt=0,\quad (A-C)\,rpdt=0,\quad (B-A)\,pqdt=0.$$

218. If *A*, *B*, *C*, be unequal, these equations will only be zero in every case when two of the quantities *p*, *q*, *r*, are zero; but then, the real axis coincides with one of the principal axes.

If two of the moments of inertia be equal, as $A=B$, the three equations are reduced to $rp=0$, $qr=0$; both of which will be satisfied, that is, they will both be zero for every value of q and p, if $r=0$. The axis of rotation is, therefore, in a plane at right angles to the third principal axis; but as the body is then a solid of revolution, every axis in that plane is a principal axis.

219. When $A=B=C$, the three preceding equations are zero, whatever may be the values of *p*, *q*, *r*, then all the axes of the body will be principal axes. Thus the principal axes alone have the property of permanent rotation, though they do not possess that property in the same degree.

220. Suppose the real axis of rotation oz'', fig. 50, to deviate by an indefinitely small quantity from oz', the third principal axis, the coefficients M and M' will then be indefinitely small, since $q = M' \times \cos(nt+g)$, and $p = M \times \sin(nt+g)$ are indefinitely small. Now if n be a real quantity, $\sin(nt+g)$, $\cos(nt+g)$, will never exceed very narrow limits, therefore q and p will remain indefinitely small; so that the real axis oz'' will make indefinitely small oscillations about the third principal axis. But if n be imaginary, by article 215, $\sin(nt+g)$, $\cos(nt+g)$, will be changed into quantities which increase with the time, and the real axis of rotation will deviate more and more from the third principal axis, so that the motion will have no stability. The value of n will decide that important point.

Since[15]

$$n = r\sqrt{\frac{(A-C)(B-C)}{AB}},$$

it will be a real quantity when C the moment of inertia with regard to oz', is either the greatest or the least of the three moments of inertia A, B, C, for then the product[16] $(A-C)(B-C)$ will be positive; but if C have a value that is between those of A and B, that product will be negative, and n imaginary. Hence the rotation will be stable about the greatest and least of the principal axes, but unstable about the third.

221. Having determined the rotation of the solid, it only remains to ascertain the position of the principal axis with regard to quiescent space, that is, with regard to the fixed axes ox, oy, oz. That evidently depends on the angles f, y, and q.

If the third principal axis oz', fig. 50, be assumed to be nearly at right angles to the plane xoy, the angle zoz', or q, will be so very small that its square may be omitted, and its cosine assumed equal to unity; then the equations (41) give $df - dy = rdt$; or if $r = a$, be a constant quantity, the integral is,

$$y = f - at + e.$$

If $\sin q \cos f = s$, $\sin q \sin f = u$, the two first of equations (41), after the elimination of dy, give

$$\frac{ds}{dt} + a\,u = -p, \quad \frac{du}{dt} - a\,s = q.$$

The integrals of these two quantities are obtained by the method in article 214, and are

$$s = x\cos(at + l) - \frac{BM'}{Ca}\cos(nt+g),$$
$$u = x\sin(at + l) - \frac{AM}{Ca}\sin(nt+g),$$

x and l being new arbitrary quantities introduced by integration. The problem is completely solved, since s and u give q and f in values of the time, and y is given in values of f and the time.

Compound Pendulums

222. Hitherto the rotation of a solid about its centre of gravity has been considered either when the body is free, or when the centre of gravity is fixed; but imagine a solid OP, fig. 54, to revolve about a fixed axis in o which does not pass through its centre of gravity. If the body be drawn aside from the vertical oz, and then left to itself, it will oscillate about that axis by the action of gravitation alone. This solid body of any form whatever is the compound pendulum, and its motion is perfectly similar to that of the simple pendulum already described, depending on the property of areas.

fig. 54.

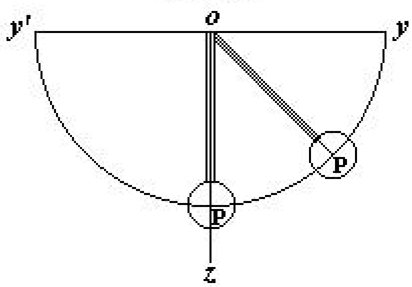
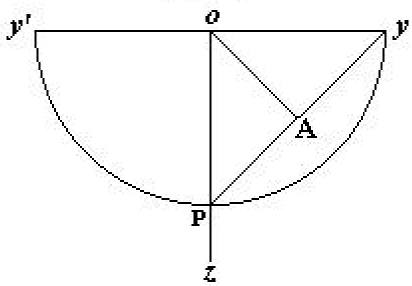


fig. 55.



The motion being in the plane zoy, the sums of the areas in the other two planes are zero; so that the motion of the pendulum is derived from the equation

$$\mathbf{S}\left(\frac{yd^2z - zd^2y}{dt^2}\right)dm = \mathbf{S}\left(y\mathrm{Z} - z\mathrm{Y}\right)dm\,.$$

In order to adapt that equation to the motion of the pendulum, let $oy = y$, $o\mathrm{P} = z$, $\mathrm{A}o = z'$, $\mathrm{A}y = y'$, hence $\mathrm{PA} = -y'$, fig.55; and let the angle PoA be represented by q. P is the centre of gravity of the pendulum, which is supposed to rotate about the axis ox, passing through o at right angles to the plane zoy, and therefore it cannot be represented in the diagram.

Now

$$-y' = z\sin q$$
$$z' = z\cos q$$
$$z' = y\cos q$$
$$y' = y\cos q$$

If the first of these four equations be multiplied by $\sin q$, and the second by $\cos q$, their sum is

$$z = z'\cos q - y'\sin q\ ;$$

in the same way

$$y = z'\sin q + y'\cos q\ .$$

If these values be substituted in the equation of areas it becomes

$$A\frac{d^2\theta}{dt^2} = -\mathbf{S}\left(y\text{Z} - z\text{Y}\right)dm\ ,$$

for

$$A = \mathbf{S}\left(y'^2 + z'^2\right)dm.$$

If the pendulum moves by the force of gravitation alone in the direction *oz*,

$$\text{Y} = 0 \qquad \text{Z} = g\ .$$

Hence

$$A\frac{d^2\theta}{dt^2} = -\mathbf{S}gydm\ .$$

If the value of *y* be substituted in this it becomes,[17]

$$A\frac{d^2\theta}{dt^2} = -g\sin\theta\ .\,\mathbf{S}\,.\,z'dm - g\cos\theta\ .\,\mathbf{S}\,.\ y'dm\ .$$

Because z' passes through the centre of gravity of the pendulum, the rotatory pressure $\mathbf{S}.y'dm$ is zero; hence

$$A\frac{d^2\theta}{dt^2} = -g\sin\theta\ .\,\mathbf{S}\,.\,z'dm\ .$$

If *L* be the distance of the centre of gravity of the pendulum from the axis of rotation ox, the rotatory pressure $\mathbf{S}\cdot z'dm$ becomes *Lm*, in which *m* is the whole mass of the pendulum; hence

$$A\frac{d^2\theta}{dt^2} = -Lmg\sin\theta\ ,$$

or

$$\frac{d\theta^2}{dt^2} = \frac{2Lmg}{A}\cdot\cos\theta + \text{C}\ ,$$

C being an arbitrary constant quantity.

223. If a simple pendulum be considered, of which all the atoms are united in a point at the distance of l from the axis of rotation ox, its rotatory inertia will be $A = ml^2$, m being the mass of the body, and $l^2 = z^2 + y^2$. In this case $l = L$. Substituting this value for *A*, we find

$$\frac{d\theta^2}{dt^2} = \frac{2g}{l}\cos\theta + \text{C}\ .$$

224. Thus it appears, that if the angular velocities of the compound and simple pendulums be equal when their centres of gravity are in the vertical, their oscillations will be exactly the same, provided also that the length of the simple pendulum be equal to the rotatory inertia of the solid body with regard to the axis of motion, divided by the product of the mass by the distance of its centre of gravity from the axis, or $l = \frac{A}{mL}$.

Thus such a relation is established between the lengths of the two pendulums, that the length of a simple pendulum may be found, whose oscillations are performed in the same time with those of a compound pendulum.

In this manner the length of the simple pendulum beating seconds has been determined from observations on the oscillations of the compound pendulum.

Notes

[1] We use boldface for **S** in this edition. Somerville uses the plain face S.

[2] The left hand side of this expression is *reads* $\left(\frac{yd^2z - zd^2y}{dt^2}\right)dm$ in the 1st edition.

[3] Segner, Johann or Jan Andreas, (1704-1777), mathematician and physicist, born in Pressburg, Hungary. Segner discovered that solid bodies have three axes of symmetry. His publications include *Elements of Arithmetic and Geometry* and *Nature of Liquid Surfaces*.

[4] See note 6, *Book I, Chapter III.*

[5] *nutation*. Oscillatory movement of the axis of a rotating body (as the earth). *Merriam-Webster's Collegiate Dictionary*.

[6] The prime in the second term in (43) is interchanged in error as $rx' - p'z = 0$ in the 1st edition.

[7] This article (201) is out of sequence. It ought to be article 199. The ordering sequence resumes at 199 in the next article. As a consequence there are two articles numbered 201. We retain the ordering followed in 1st edition.

[8] There are two articles numbered 201 as noted above.

[9] "Equation (46)" *reads* "This equation" in the 1st edition (published erratum).

[10] The 1st edition has a period after this expression.

[11] The principle of logarithms was devised by John Napier (1550-1617) of Merchiston, Scotland, to abridge arithmetical calculations, by the use of addition and subtraction in place of multiplication and division.

[12] Written with a rounded right-hand bracket in 1st edition.

[13] Punctuation in equation changed to period from semicolon in 1st edition.

[14] Comma added.

[15] This *reads* $n = r\sqrt{\frac{(A-B)(B-C)}{AB}}$ in the 1st edition. But from the analysis in article 216, the first factor in the numerator under the root should be $(A-C)$ not $(A-B)$.

[16] As in the 1st edition (see preceding note.)

[17] The first right hand term in this expression *reads* $-g\sin q\ \text{S}\cdot z'dm$ in 1st edition.